Comment on "A dynamic nonlinear subgrid-scale stress model" [Phys. Fluids **17**, 035109 (2005)]

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In a recent paper Wang and Bergstrom introduce a dynamic nonlinear model for the deviatoric part τ_{ij}^* of the subgrid-scale (SGS) stress.[1] In terms of their notation this model may be written as

$$\tau_{ij}^* = \tau_{ij}^{*(1)} + \frac{2C_W}{C_S|\bar{S}|}P_{ij}^{(1)} + D_{ij}^{(1)}. \tag{1}$$

The first-order approximations of the SGS stress tensor $\tau_{ij}^{*(1)}$, production $P_{ij}^{(1)}$ of stress by shear, and production source term $D_{ij}^{(1)}$ are given by the expressions

$$\tau_{ij}^{*(1)} = -2\nu_{\text{SGS}}\bar{S}_{ij}, \tag{2a}$$

$$P_{ij}^{(1)} = -\bar{u}_{i,k}\tau_{kj}^{*(1)} - \bar{u}_{j,k}\tau_{ki}^{*(1)} + \frac{2}{3}\bar{S}_{mn}\tau_{nm}^{*(1)}\delta_{ij}, \tag{2b}$$

$$D_{ij}^{(1)} = \frac{C_N + 2C_W}{C_S|\bar{S}|}\left(\bar{S}_{ik}\tau_{kj}^{*(1)} + \bar{S}_{jk}\tau_{ki}^{*(1)} - \frac{2}{3}\bar{S}_{mn}\tau_{nm}^{*(1)}\delta_{ij}\right). \tag{2c}$$

In these expressions, $\bar{S}_{ij} = (\bar{u}_{i,j} + \bar{u}_{j,i})/2$ is the filtered rate-of-strain tensor, and $|\bar{S}| = (2\bar{S}_{ik}\bar{S}_{ki})^{1/2}$ is a characteristic strain rate. The SGS viscosity $\nu_{\text{SGS}} = C_S\bar{\Delta}^2|\bar{S}|$, where $\bar{\Delta}$ refers to the length scale associated with the filter size. To simplify the comparisons presented below the last two terms of (1) are written here in terms of C_S and $|\bar{S}|$, but the use of $\nu_{\text{SGS}} = C_S\bar{\Delta}^2|\bar{S}|$ in (1) and (2a)–(2c) reveals that these two terms are actually independent of C_S and $|\bar{S}|$. By adopting a methodology suggested by Wong,[2] the model parameters C_S, C_W, and C_N are calculated such that the local error function is minimized. The latter constraint implies the following linear equation system for C_S, C_W, and C_N (see the corresponding definitions of Wang and Bergstrom):[1]

$$\begin{bmatrix} M_{ij}M_{ij} & M_{ij}W_{ij} & M_{ij}N_{ij} \\ W_{ij}M_{ij} & W_{ij}W_{ij} & W_{ij}N_{ij} \\ N_{ij}M_{ij} & N_{ij}W_{ij} & N_{ij}N_{ij} \end{bmatrix}\begin{bmatrix} C_S \\ C_W \\ C_N \end{bmatrix} = -\begin{bmatrix} \mathcal{L}_{ij}^*M_{ij} \\ \mathcal{L}_{ij}^*W_{ij} \\ \mathcal{L}_{ij}^*N_{ij} \end{bmatrix}. \tag{3}$$

Solutions for the model parameters C_S, C_W, and C_N exist if and only if the three differential tensorial functions M_{ij}, W_{ij}, and N_{ij} are linearly independent. The authors suspect that these three different tensorial functions are, in fact, independent. However, a strict analytical proof is not yet available to support such a conjecture, and it remains an open question for further analytical explorations.[1] Regarding the suitability of their dynamic nonlinear model (DNM) (19) [which is relation (1) here] as SGS stress tensor model the authors conclude lastly, the proposed DNM formula (19) is, at most, an assumption for modeling the SGS stress. Whether the right- and left-hand sides of formula (19) are satisfactorily correlated can be examined via possible future *a priori* approaches based on DNS and experimental data analysis.[1]

The latter question will be addressed here. One may distinguish two approaches to construct SGS stress tensor models. One way (the approach of Wang and Bergstrom) is to parametrize this tensor directly by adopting heuristic assumptions. Models obtained in this way are not shown to be correct; one needs direct numerical simulation (DNS) or experimental data to demonstrate the suitability of the assumptions applied. The conclusions obtained by such comparisons then depend on the specific flow configuration considered, and the Reynolds number and grid spacing applied. In addition to that, the way to construct such models does not prove the general existence of solutions to such model equations. An alternative to the construction of heuristic SGS stress tensor models is to explain the underlying physics. The SGS stress tensor is the result of the correlations of SGS velocity fluctuations. One needs, therefore, a physically correct model for dynamics of SGS velocity fluctuations in order to derive then the SGS stress tensor as a consequence. The latter approach has some obvious advantages compared to the application of heuristic SGS stress tensor models. First, the general existence of solutions to model equations for SGS velocity fluctuations proves the general existence of solutions to filtered velocity equations. Second, there are first principles available for the construction of physically consistent models for SGS velocity fluctuations,[3] which provides support for the proof of the suitability of such models. Third, the consistency between underlying equations for SGS fluctuations and SGS stress tensor models (this means the existence of a systematic hierarchy of simple and more complex models) is helpful for the understanding of the range of applicability of algebraic SGS stress tensor models. It provides alternative methods (transport equations for the stress tensor and velocity fluctuations) which may be applied under conditions where the use of algebraic SGS stress tensor models appears to be questionable, and it enables the development of consistent hybrid methods for velocity and reacting scalar

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fields.[3] SGS velocity models that are required to apply this approach were developed recently. They were shown to have a consistent theoretical basis, and their predictions agree well with the DNS data and experimental results.[3–7] By following a recent publication of the author,[7] their implications for the SGS stress tensor will be shown here in order to address the suitability of Wang and Bergstrom's SGS stress model (1)–(3).

By adopting the simplified Langevin model for SGS velocity dynamics, one may derive the following equation for the deviatoric part of the SGS stress tensor in incompressible flows:[7]

$$\frac{\partial \tau_{ij}^*}{\partial t} + \bar{u}_k \frac{\partial \tau_{ij}^*}{\partial x_k} + \frac{\partial T_{ijk}}{\partial x_k} = P_{ij} - 4\frac{\nu_r}{\tau_L}\bar{S}_{ij} - \frac{2}{\tau_L}\tau_{ij}^*. \quad (4)$$

Here, T_{ijk} represents contributions due to triple correlations, and

$$P_{ij} = -\bar{u}_{i,k}\tau_{kj}^* - \bar{u}_{j,k}\tau_{ki}^* + \frac{2}{3}\bar{S}_{mn}\tau_{nm}^*\delta_{ij} \quad (5)$$

represents the production of stress by shear. The residual viscosity is given by $\nu_r = k_r\tau_L/3$, where k_r denotes the residual turbulent kinetic energy and τ_L refers to the characteristic time scale of velocity fluctuations. By assuming a local equilibrium between production and dissipation [by neglecting the left-hand side of (4)] we obtain

$$\tau_{ij}^* = -2\nu_r\bar{S}_{ij} + \frac{\tau_L}{2}P_{ij}. \quad (6)$$

In the second order of approximation, relation (6) implies the nonlinear SGS stress tensor model derived by the author,[7]

$$\tau_{ij}^* = \tau_{ij}^{*(1)} + \frac{2C_W}{C_S|\bar{S}|}P_{ij}^{(1)}. \quad (7)$$

Here, $\tau_{ij}^{*(1)}$ and $P_{ij}^{(1)}$ are given by (2a) and (2b); this means $\nu_{SGS} = \nu_r$. To simplify the model comparison below, τ_L is taken in (7) with reference to $|\bar{S}|$: we assumed that $\tau_L = 4C_W/(C_S|\bar{S}|)$. The model parameters in (7) can be obtained by adopting a dynamic procedure.[7] In terms of the notation used by Wang and Bergstrom the corresponding equations are given by

$$\begin{bmatrix} M_{ij}M_{ij} & M_{ij}A_{ij} \\ A_{ij}M_{ij} & A_{ij}A_{ij} \end{bmatrix}\begin{bmatrix} C_S \\ C_W \end{bmatrix} = -\begin{bmatrix} \mathcal{L}_{ij}^*M_{ij} \\ \mathcal{L}_{ij}^*A_{ij} \end{bmatrix}, \quad (8)$$

where $A_{ij} = W_{ij} - 2N_{ij}$ is introduced. The solutions of (8) always exist since M_{ij} and A_{ij} are linearly independent: A_{ij} involves the filtered rate-of-rotation tensor $\bar{\Omega}_{ij}$ whereas M_{ij} is independent of $\bar{\Omega}_{ij}$.

The models (1)–(3) of Wang and Bergstrom and (7) and (8) of Heinz differ by the appearance of $D_{ij}^{(1)}$ in (1) [which is nonzero in general: see (3)]. What are the advantages and disadvantages of considering a nonzero $D_{ij}^{(1)}$? One may assume that it is an advantage to have the additional adjustable parameter C_N (a more flexible method) available. However, one has to pay for that a price due to the fact that physically consistent dynamics of velocity fluctuations with correlations

as specified by (1) [this means with a nonzero $D_{ij}^{(1)}$] do not exist [see the derivation of (7)]. First, the general existence of solutions to corresponding filtered velocity equations cannot be shown by demonstrating the general existence of underlying velocity fluctuations: it is then at least unclear whether such solutions do exist in general. Second, realizability (the constraint that any moment and correlation transport equations considered should represent realizable equations for moments and correlations of a stochastic process) was proved to represent a valuable guiding principle for turbulence modeling.[8–11] Correspondingly, the use of realizable models was found to be of remarkable relevance regarding the application of probability density function (PDF) and Reynolds-averaged Navier-Stokes (RANS) methods.[12–14] A corresponding importance of realizable models has to be expected with regard to the accuracy of models for dynamics of SGS variables. Third, inconsistencies of SGS stress tensor models and models for SGS fluctuations are not helpful for the use of SGS stress tensor models as part of more general methods (hybrid methods for velocity and reacting scalar fields).

The three disadvantages of the model (1)–(3) described above can be avoided by applying the model (7) and (8). The use of (7) and (8) represents, therefore, at least a valid alternative to the use of (1)–(3). Obviously, a more complete insight into the differences between these two models requires extensive comparisons of their performance with regard to simulations of various flows.

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