Homework 5 (MATH 5490-01) **Name (Print):**
Due date: Thursday, May 2, 2013

Consider the following stochastic velocity model for positions x_i^* and velocities V_i^*

$$\frac{dx_i^*}{dt} = V_i^*, \tag{1a}$$

$$\frac{dV_i^*}{dt} = -\frac{V_i^* - U_i}{\tau} + \sqrt{\frac{4e}{3\tau}}\frac{dW_i}{dt}. \tag{1b}$$

Here, dW_i / dt is the derivative of a Wiener process (i = 1, 3). The model involves three parameters: U_i is the mean velocity, e is the specific kinetic energy (it has the dimension of a squared velocity), and τ is the characteristic time scale of velocity fluctuations. All the three parameters may depend on time t and the position $\mathbf{x}^*(t)$.

1. Assume that the positions x_i^* and velocities V_i^* are combined to a six-dimensional vector $\mathbf{Z} = (\mathbf{x}^*, \mathbf{V}^*)$. In matrix notation, Eqs. (1) can be written

$$\frac{d\mathbf{Z}}{dt} = \mathbf{a} + \mathbf{G}(\mathbf{Z} - \langle \mathbf{Z} \rangle) + \mathbf{b}\frac{d\mathbf{W}}{dt}.$$

   Here, **a** is a six-dimensional vector, and **G** and **b** are 6 × 6 matrices.
   a) Specify **a**, **G**, and **b** according to the equation system (1).
   b) What are the requirements for the coefficients **a, G**, and **b** under which the analytical solution approach for Fokker-Planck equations can be used for the calculation of the velocity-position joint PDF f(**w**, **x**, t) related to Eqs. (1)?

2. The asymptotic change dV_i^* / dt can be considered to be small compared to the right-hand side of Eq. (1b). The asymptotic velocities $V_i^* = dx_i^* / dt$ are described for this case by the equation (it is assumed that U_i, e, and τ are constants)

$$\frac{dx_i^*}{dt} = U_i + \sqrt{\frac{4}{3}e\tau}\frac{dW_i}{dt}.$$

   The position PDF f(**x**, t), which is related to $\mathbf{x}^*(t)$, and the conditional PDF f(**x**, t | **x'**, t') are related by $f(\mathbf{x}, t) = \int f(\mathbf{x}, t \mid \mathbf{x}', t')\, f(\mathbf{x}', t')\, d\mathbf{x}'$.
   a) Determine f(**x**, t | **x'**, t') by applying the analytical solution approach for a Fokker-Planck equation. Simplify f(**x**, t | **x'**, t') as much as possible by using the expressions obtained for the parameters of f(**x**, t | **x'**, t').
   b) Determine the asymptotic conditional PDF f(**x**, t | **x'**, t') as $t \to \infty$.
   c) Calculate the corresponding asymptotic position PDF f(**x**, t).