Math 2205 Calculus II — Exam I
Fall 2013 — Thursday, Sept 19, 2013

Instructions:
• Show all your work, using the space provided on the exam. Use correct mathematical notation.
• Clearly mark your solution by circling it.
• A basic scientific calculator is allowed. No graphing calculator is allowed, and cell phones may not be used during the exam.
• Present your Photo I.D. when turning in your exam
• The exam has 6 pages. Please check to see that your copy has all the pages.

1. (12 points) Find the area of the region enclosed by the line $y = x$ and the parabola $x = y^2 - 2$. Simplify your answer.

Intersect: $y = y^2 - 2$

$y^2 - y - 2 = 0$

$(y - 2)(y + 1) = 0\Rightarrow y = 2, -1$

\[
A = \int_{-1}^{2} \left( y^2 - 2 \right) dy 
\]

\[
A = \left[ \frac{1}{3} y^3 - 2y \right]_{-1}^{2} 
\]

\[
= \left( \frac{8}{3} - \frac{4}{3} \right) - \left( \frac{1}{3} - \frac{1}{3} - 2 \right) 
\]

\[
= \frac{2}{3} + 4 - \frac{1}{3} - 2 = \frac{4}{3} 
\]

$A = \frac{4}{3}$
2. (10 points) Find the average value of the function \( f(x) = x\sqrt{x^2 - 1} \) on \([1, 3]\).

\[
\bar{f} = \frac{\int_1^3 x\sqrt{x^2 - 1} \, dx}{3 - 1}
= \int_1^3 \frac{x\sqrt{x^2 - 1}}{2} \, dx
= \int_0^8 u^{\frac{1}{2}} \left( \frac{1}{2} \, du \right)
= \frac{1}{4} \int_0^8 u^{\frac{1}{2}} \, du
= \frac{1}{4} \left( \frac{2}{3} u^{\frac{3}{2}} \right)_0^8
= \frac{1}{6} \cdot 8^{\frac{3}{2}}
= \frac{8}{3}\sqrt{2}
\]

3. (6 points) Find the average value of \( g(x) \) on \([0, 8]\) where the graph of \( y = g(x) \) is shown below.

\[
\bar{g} = \frac{\int_0^8 g(x) \, dx}{8 - 0} = \frac{-\frac{1}{2} + 1 + 1 + \frac{3}{2} + 2}{8}
= \frac{9}{8}
\]
4. (12 points) The base of a solid \( S \) is the region enclosed by the parabola \( y = 9 - x^2 \) and the \( x \)-axis. Cross sections perpendicular to the \( y \)-axis are rectangles with height equal to one-half of the width. Calculate the volume of the solid \( S \).

\[
V_S = 2x^2 \, dy = 2(9-y) \, dy
\]

\[
TV = \int_0^9 2(9-y) \, dy = 2 \int_0^9 (9-y) \, dy
\]

\[
u = 9 - y \quad y = 0 \quad u = 9
\]

\[
du = -dy
\]

\[
\Rightarrow \, dy = -du
\]

\[
= 2 \int_9^0 u \, (-du) = 2 \int_0^1 u \, du = 2 \left[ \frac{1}{2}u^2 \right]_0^9 = 9^2 - 0 = 81
\]

5. (10 points) Consider the solid generated by rotating the region bounded by \( x = 0 \) and \( x = -y^2 + 4y - 3 \) (see the diagram) about the \( y \)-axis. Set up an integral, in terms of a single variable, that gives the volume of the solid. DO NOT EVALUATE THE INTEGRAL.

Intercepts: \((-y^2 - 4y + 3) = -(y-3)(y-1) = 0\)

\( y \) = 3, 1

\[
r = -y^2 + 4y - 3
\]

\[
V_o = \pi \int r^2 \, dy = \pi \int (4y - y^2 - 3)^2 \, dy
\]

\[
TV = \int_1^3 V_o = \int_1^3 \pi \left( (4y - y^2 - 3)^2 \right) \, dy
\]
6. (10 points) Consider the solid generated by rotating the region bounded by the curves \( y = x^4 \) and \( y = x \) about the line \( x = 3 \). Set up an integral, in terms of a single variable, that gives the volume of the solid. DO NOT EVALUATE THE INTEGRAL.

\[
V_S = 2\pi \int_0^1 (3-x)\,(x-x^4)\,dx
\]

\[
TV = \int_0^1 2\pi (3-x)\,(x-x^4)\,dx
\]

7. (10 points) A force of 10lb is required to hold a spring stretched to 4in beyond its natural length. How much work is done in stretching the spring from its natural length to 6in beyond its natural length?

\[
F = kx \quad \rightarrow \quad 10\,lbs = k \left( \frac{4}{12} \, \text{ft} \right) = k \left( \frac{1}{3} \, \text{ft} \right) \quad \rightarrow \quad k = 30 \, \text{lbs/ft}
\]

\[
F_0 = 30x \, \text{lbs}
\]

\[
D_0 = dx \, \text{ft}
\]

\[
W_0 = 30x \, dx \, \text{ft-lbs}
\]

\[
TW = \int_0^{1/2} W_0 = \int_0^{1/2} 30x \, dx = 15x^2 \bigg|_0^{1/2} = 15 \left( \frac{1}{4} \right)^2 - 0 = \frac{15}{4} \, \text{ft-lbs}
\]
8. (10 points) A spherical tank of radius 10 ft is buried underground. The highest point on the surface of the tank is 4 ft below ground level. If the tank is half full with a fluid that has a density of 50 lbs/ft³, write an integral that describes the work required to pump all of the fluid to ground level. Be sure to include a clear diagram which specifically indicates your choice of variable. DO NOT EVALUATE THE INTEGRAL.

\[ W = \int_0^{10} \omega_s = \int_0^{10} 50 \pi (100 - x^2)(x+14) \, dx \quad \text{ft-lbs} \]
9. (8 points) Evaluate \( \int_1^3 y \ln y \, dy \).

\[
\int_1^3 y \ln y \, dy = \frac{1}{2} y^2 \ln y \bigg|_1^3 - \frac{1}{2} \int_1^3 y \, dy
\]

\[
u = \ln y \quad dv = \frac{dy}{y} \quad \frac{du}{dy} = \frac{1}{y} \quad v = \frac{y^2}{2} \]

\[
= \frac{1}{2} y^2 \ln y \bigg|_1^3 - \frac{1}{4} y^2 \bigg|_1^3
\]

\[
= \left( \frac{9}{2} \ln(3) - \frac{7}{4} \right) - \left( 0 - \frac{1}{4} \right)
\]

\[
= \frac{9}{2} \ln(3) - 2
\]

10. (12 points) Evaluate \( \int x^2 e^x \, dx \).

\[
\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx
\]

\[
u = x^2 \quad dv = e^x \, dx \quad du = 2x \, dx \quad v = e^x
\]

\[
= x^2 e^x - 2 \left( xe^x - \int e^x \, dx \right)
\]

\[
= x^2 e^x - 2xe^x + 2e^x + C
\]

For Instructor Use Only:

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