Hyperbolic Geometry

We started by defining the it angle of parallelism to be the smallest angle giving a ray parallel to a line.

We proved that \( d_0 \leq 90^\circ \). If \( d_0 = 90^\circ \), then we have Euclidean Geometry, if \( d_0 < 90^\circ \), then we have Hyperbolic Geometry.

This can be used to find two lines through a point parallel to another.

We can also then deduce that the summit angles of a Saccheri Quadrilateral have measure \(< 90^\circ\), which tells us that rectangles do not exist in hyperbolic geometry. This leads us to parallel lines not being everywhere equidistant and therefore the summit of a Saccheri Quadrilateral is longer than the base.

The defect of a triangle is "how much less" than \( 180^\circ \) the degree measure of the triangle is.

We could prove that in hyperbolic geometry, the defect of a triangle is positive, and therefore the angle sum of the triangle is less than \( 180^\circ \).

Not all triangles have the same defect since the defect of triangles, and even general polygons, is additive.

Triangles having non-equal defects allowed us to state the Angle-Angle-Angle congruence theorem for triangles in hyperbolic geometry. Having all angle congruent gives us equal defects, and congruent triangles.

In Euclidean geometry, all discussions of area stemmed from

**Postulate 26:** The area of a rectangle is the product of the base times the height. Without rectangles in hyperbolic geometry, we need a new definition of area.

We tied the idea of area to the defect of a polygon.

**Definition:** For any polygon \( P \), the area of \( P \) is directly proportional to the defect of the polygon. That is, \( A(P) = k \cdot d(P) \) where \( k \) is some positive constant.

This gave us 1) each polygon has only one area, 2) congruent polygons have equal area, and 3) areas are additive.
The model we used for hyperbolic geometry was the *Poincaré Disk*, the points inside the unit circle.

We defined lines as diameters or orthogonal arcs.

We had to re-define distance because Postulate 3 tells us that if we select a real number, there exist two points that distance apart. In the disk with conventional distance, we would be limited to distances less than 2. If you are asked to find hyperbolic distance, I will give you the formula.

**Projective Geometry**

The axiom system for the projective plane.

No such thing as parallel lines

Infinite model: Real projective plane - where do parallel line meet? What has to be added?

Infinite model: Order of a finite projective plane, go over the Fano plane.

What is the Dual problem? Is there a dual for Euclidean or Hyperbolic geometry?