(1) A famous old problem is to find a polynomial function \( f(x) \) so that for each positive integer \( n = 1, 2, 3, \ldots \), results in \( f(n) \) being a prime. A good candidate is \( f(x) = x^2 + x + 41 \). Find and factor the first three integers which, when substituted into this polynomial, yield composites.

(2) This is one of the most troubling unsolved problems in mathematics: "Any even number greater than or equal to 4 can be written as the sum of two primes."

(a) Is the conjecture true if we replace "even" with "odd"?

(b) Is this conjecture true if we replace "sum" by "product"?

(3) Clearly, any positive integer must assume one of the six forms:

\[
6k, 6k + 1, 6k + 2, 6k + 3, 6k + 4, 6k + 5
\]

(a) Use this to show that any prime greater than 3 is either on more or five more that a multiple of 6.

(b) Now suppose we say that if \( p, p + 2 \), and \( p + 4 \) are all primes, we’ll call them "triplet primes". While it is unknown whether there are finitely or infinitely many pairs of prime twins, we can determine whether the following conjecture is true of false using (a):

(i) There are infinitely many prime triplets

(4) In 1848, DePolignac asserted: "Every odd number can be expressed as the sum of a power of 2 and a prime." Find a counter-example to this assertion.

(5) A famous result from number theory states: "Any prime of the form \( 4n + 1 \) may be written as the sum of two squares in one and only one way, whereas a prime of the form \( 4n + 3 \) may never be written as the sum of two squares." For example, \( 4(3) + 1 = 13 \) and \( 13 = 2^2 + 3^2 \), but \( 4(2) + 3 = 11 \), but \( 8 \neq a^2 + b^2 \) for any \( a \) or \( b \). This first part of this (originally stated by Fermat, but proved by Euler) is difficult, but the second, the \( 4n + 3 \) case isn’t too bad. Prove it. (HINT: consider three cases, where \( a \) and \( b \) are both even, where \( a \) and \( b \) are both odd, and where \( a \) is even and \( b \) is odd.)