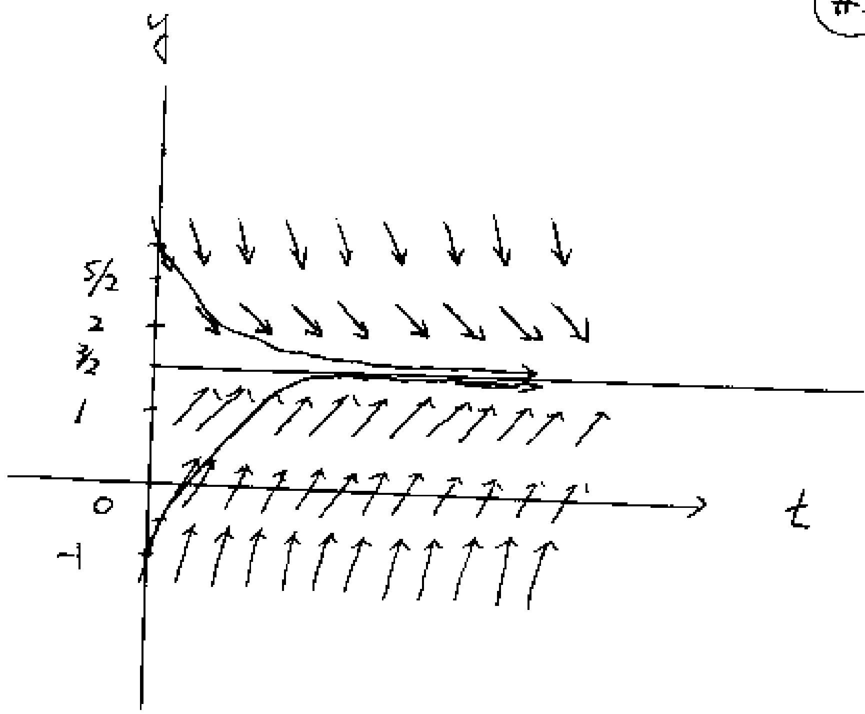
§1.1

#1 $y' = 3 - 2y$

(t, y)	$f(t,y) = 3-2y$
$(t, \frac{5}{2})$	-2
$(t, 2)$	-1
$(t, 3/2)$	0
$(t, 1)$	1
$(t, 0)$	3
$(t, -1)$	5



$y \to 3/2$ as $t \to \infty$

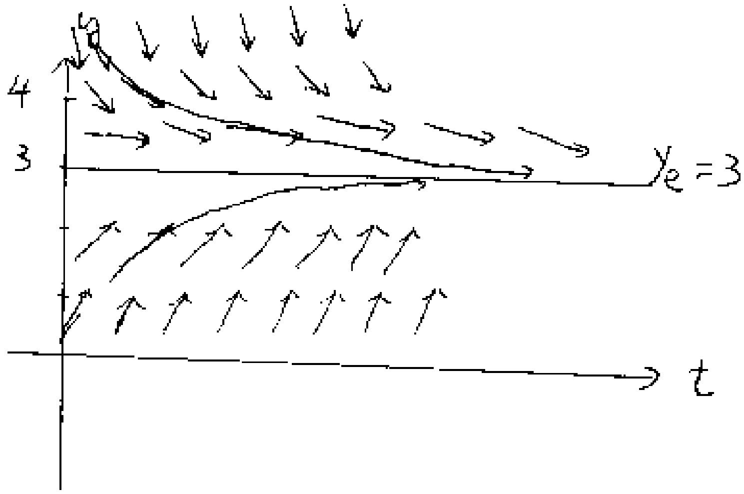
The above behavior does not depend on the initial value of y at $t=0$

7

$$\frac{dy}{dt} = ay + b$$

$$y(t) = -\frac{b}{a} + Ce^{at}$$

$y(t) \to -\frac{b}{a}$ if $a < 0$.



since all solution approach $y = 3 = -\frac{3}{-1} = -\frac{b}{a}$

$a = -1, \quad b = 3$

$\Rightarrow \boxed{\frac{dy}{dt} = 3 - y}$

(using the approach of direction fields is also fine)
(slope)

#24 Let q be the drug that is present in the bloodstream

(a)

$$\frac{dq}{dt} = \underbrace{5\ \frac{mg}{cm^3} \cdot 100 \cdot \frac{cm}{hr}}_{\text{drug enters blood stream}} - \underbrace{0.4\ \frac{1}{hr}}_{\text{absorbing rate}} \cdot q$$

$$\Rightarrow \quad \frac{dq}{dt} = 500 - 0.4q = b + aq, \quad a = -0.4, \quad b = 500$$

$$q \to \underbrace{-\frac{b}{a}}_{\substack{\text{equilibrium} \\ \text{solution:} \\ f(t,y)=0.}} = -\frac{500}{-0.4} = 1250 \text{ mg} \quad \text{as } t \to \infty$$

§ 1.2

#1. (a)

$$\frac{dy}{dt} = -y+5$$

$$\frac{1}{-y+5}\, dy = 1$$

$$\int \frac{1}{-y+5} \left(\frac{dy}{dt}\right) dt = \int dt$$

Let $u = -y+5 \quad -du = dy$

$$-\int \frac{du}{u} = \int dt$$

$$\ln|u| = -t + c$$

$$-y+5 = ce^{-t}$$

$$y = 5 + ce^{-t}$$

$$y(0) = y_0 \Rightarrow 5 + c = y_0 \Rightarrow c = y_0 - 5$$

$$\boxed{y(t) = 5 + (y_0 - 5)e^{-t}}$$

#4

(a) $\quad a y_e - b = 0 \quad , \quad y_e = \frac{b}{a}$

(b) $\quad Y(t) = y - y_e \quad$ where $\quad y_e = \frac{b}{a}$

$$\frac{dY(t)}{dt} = \frac{dy}{dt} - \cancel{\frac{dy_e}{dt}}^{\,=0}$$

$$= ay - b$$

$$= a(Y(t) + y_e) - b$$

$$= aY(t) + \underbrace{ay_e - b}_{=0}$$

$$\Rightarrow \boxed{\frac{dY(t)}{dt} = aY(t)}$$

#7

(a) $P(t) = 900 + ce^{t/2}$

$P(0) = 850 \Rightarrow 900 + c = 850, \quad c = -50$

$\Rightarrow P(t) = 900 - 50e^{t/2}$

when the population becomes extinct,

$P(t) = 0 \Rightarrow 900 - 50e^{t/2} = 0$

$e^{t/2} = 18$

$t/2 = \ln(18)$

$t = 2 \cdot \ln(18) \approx 5.78$ months

(b) $P(t) = 900 + ce^{t/2}$

$P(0) = 900 + c = P_0 \Rightarrow c = P_0 - 900$

$P(t) = 900 + \underbrace{(P_0 - 900)}_{<0} e^{t/2}$

$P(t) = 0 \Rightarrow 900 = (900 - P_0)e^{t/2}$

$\frac{900}{900 - P_0} = e^{t/2}$

$t = 2\ln\left(\frac{900}{900 - P_0}\right)$

(c)

$$P(t) = 900 + (P_0 - 900)e^{t/2}$$

One year = 12 months

$$P(12) = 0 \Rightarrow 900 + (P_0 - 900)e^6 = 0$$

$$900/e^6 = 900 - P_0$$

$$\Rightarrow P_0 = 900 - 900/e^6$$

$$= 900(1 - e^{-6})$$

$$\approx 897.8$$

#9

(a)
$$\frac{dv}{dt} = 9.8 - v/5 = b + av,$$

where $b = 9.8$, $a = -\frac{1}{5} = -0.2$

$$v(t) = -\frac{b}{a} + ce^{at}$$

$$= \frac{9.8}{0.2} + ce^{-0.2t} = 49 + ce^{-0.2t}$$

$v(0) = 0 \Rightarrow 49 + c = 0 \quad c = -49$

Hence
$$v(t) = 49 - 49e^{-0.2t}$$

$t \to \infty$ $v(t) = 49$, the limiting velocity

98% of $49 = (0.98) \cdot 49$

$$= 49(1 - e^{-0.2t})$$

$$\Rightarrow \quad 1 - e^{-0.2t} = 0.98$$

$$0.02 = e^{-0.2t}$$

$$t = \frac{\ln(0.02)}{-0.2} = -5\ln\left(\frac{1}{50}\right) = 5\ln 50 = 19.56 \text{ sec.}$$

(b)

$$\frac{dx}{dt} = 49(1 - e^{-0.2t}) \quad , \quad x(0) = 0$$

$$x = 49t + 245e^{-0.2t} - 245$$

$$x(5\ln 50) = 245\ln 50 + 245 \cdot \frac{1}{50} - 245$$

$$= 245\left(\ln 50 + \frac{1}{50} - 1\right)$$

$$= 718.3456$$

§ 1.3

#8

$$\left.\begin{aligned} y_1(t) &= e^{-3t} \\ y_1'(t) &= -3e^{-3t} \\ y_1''(t) &= 9e^{-3t} \end{aligned}\right\} \quad y_1'' + 2y_1' - 3y_1 = 9e^{-3t} - 6e^{-3t} - 3e^{-3t} = 0$$

#4

$$\left.\begin{aligned} y_2(t) &= e^t \\ y_2'(t) &= e^t \\ y_2''(t) &= e^t \end{aligned}\right\} \quad \begin{aligned} &y_2'' + 2y_2' - 3y_2 \\ &= e^t + 2e^t - 3e^t \\ &= 0 \end{aligned}$$

#17

$$y(t) = e^{rt}$$

$$y'(t) = re^{rt}$$

$$y''(t) = r^2 e^{rt}$$

$$\begin{aligned} y'' + y' - 6y &= r^2 e^{rt} + re^{rt} - 6e^{rt} \\ &= e^{rt}(r^2 + r - 6) \\ &= 0 \end{aligned}$$

$\Rightarrow \quad r^2 + r - 6 = 0 \quad$ (since $e^{rt} \neq 0$ for all t)

$\Rightarrow \quad (r+3)(r-2) = 0 \qquad r = 2, -3$

#20

$$\left.\begin{aligned} y(t) &= t^r \\ y'(t) &= rt^{r-1} \\ y''(t) &= r(r-1)t^{r-2} \end{aligned}\right\} \quad \begin{aligned} &t^2 y'' - 4ty' + 4y \\ &= r(r-1)t^r - 4rt^r + 4t^r \\ &= t^r(r(r-1) - 4r + 4) \\ &= 0 \end{aligned}$$

since $t^r \neq 0$ for $t > 0$

$$r(r-1) - 4r + 4 = 0$$

or $$r^2 - 5r + 4 = 0$$

$$(r-1)(r-4) = 0$$

$$r = 1, 4.$$