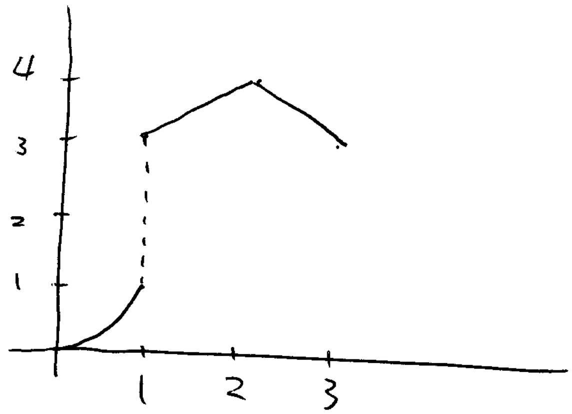
HW #11 key

#1

$$f(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ 2+t & 1 < t \le 2 \\ 6-t & 2 < t \le 3 \end{cases}$$



f is piecewise continuous by the graph.

#5

(a)

$$\int_0^\infty e^{-st} t\,dt = \lim_{A\to\infty} \int_0^A e^{-st} t\,dt$$

integration by parts

$$= \lim_{A\to\infty} t \cdot \frac{-1}{s} e^{-st} \Big|_0^A - \lim_{A\to\infty} \int_0^A \frac{-1}{s} e^{-st}\,dt$$

$$= \frac{-1}{s^2} \lim_{A\to\infty} e^{-st} \Big|_0^A = \frac{1}{s^2}$$

(b)

$$\int_0^\infty e^{-st} t^2\,dt = \lim_{A\to\infty} \int_0^A e^{-st} t^2\,dt$$

integration by parts

$$= \lim_{A\to\infty} t^2 \frac{-1}{s} e^{-st} + \frac{2}{s} \lim_{A\to\infty} \int_0^A e^{-st} t\,dt$$

from (5)(a)

$$= \frac{2}{s} \cdot \frac{1}{s^2} = \frac{2}{s^3} = \frac{2!}{s^{2+1}}$$

(c)

By ~~[illegible]~~ mathematical induction, we claim that

$$\int_0^\infty e^{-st} t^n dt = \frac{n!}{s^{n+1}} \qquad (1)$$

Pf:

When $n=1$ & $n=2$, (1) is true, by 5(a) & 5(b)

Assume $n=k$ is also true

We will show that $n=k+1$ is true.

Since

$$\int_0^\infty e^{-st} t^{k+1} dt = \lim_{A\to\infty} \int_0^A e^{-st} t^{k+1} dt$$

integration by parts

$$= \lim_{A\to\infty} t^{k+1} \frac{-1}{s} e^{-st} \Big|_0^A + \frac{k+1}{s} \lim_{A\to\infty} \int_0^A e^{-st} t^{k} dt$$

$$= \frac{k+1}{s} \cdot \frac{k!}{s^{k+1}}$$

$$= \frac{(k+1)!}{s^{(k+1)+1}},$$

(1) is true for $n=k+1$

Thus we confirm that $\int_0^\infty e^{-st} t^n dt = \frac{n!}{s^{n+1}}$ by mathematical induction.

#6 $\mathcal{L}\{\cos at\}$

$$= \int_0^\infty e^{-st}\cos at\, dt$$

$$= \lim_{A\to\infty} \int_0^A e^{-st}\cos at\, dt$$

$$= \lim_{A\to\infty} \left[\frac{e^{-st}\sin at}{a}\Big|_0^A + \frac{s}{a}\int_0^A e^{-st}\sin at\, dt \right]$$

$$= \lim_{A\to\infty} \left[\frac{s}{a} \int_0^A e^{-st}\sin at\, dt \right]$$

$$= \frac{s}{a} \lim_{A\to\infty} \int_0^A e^{-st}\sin at\, dt$$

by example 6 on page 311

$$= \frac{s}{a} \cdot \frac{a}{s^2+a^2} = \frac{s}{s^2+a^2} \qquad s>0$$

#11 $\sin bt = \dfrac{e^{ibt} - e^{-ibt}}{2i}$

$$\int_0^\infty e^{-st}\sin bt\, dt = \frac{1}{2i}\left[\int_0^\infty e^{-st} e^{+ibt}\,dt - \int_0^\infty e^{-st}e^{-ibt}\,dt\right]$$

$$= \frac{1}{2i}\left[\mathcal{L}\{e^{ibt}\} - \mathcal{L}\{e^{-ibt}\}\right]$$

by example 5

$$\overset{\downarrow}{=} \frac{1}{2i}\left[\frac{1}{s-ib} - \frac{1}{s+ib}\right]$$

$$= \frac{1}{2i}\left[\frac{s+ib-s+ib}{s^2+b^2}\right]$$

$$= \frac{b}{s^2+b^2}$$

\#12 $\cos bt = \dfrac{e^{ibt}+e^{-ibt}}{2}$

$$\int_0^\infty e^{-st}\cos bt\,dt = \frac{1}{2}\left[\int_0^\infty e^{-st}e^{ibt}\,dt + \int_0^\infty e^{-st}e^{-ibt}\,dt\right]$$

$$= \frac{1}{2}\left[\mathcal{L}\{e^{ibt}\} + \mathcal{L}\{e^{-ibt}\}\right]$$

$$= \frac{1}{2}\left(\frac{1}{s-ib} + \frac{1}{s+ib}\right)$$

$$= \frac{1}{2}\,\frac{2s}{s^2+b^2}$$

$$= \frac{s}{s^2+b^2}$$

#3 $F(s)=\dfrac{2}{s^2+3s-4} = \dfrac{2}{(s+4)(s-1)}$

$$= \frac{a}{s+4} + \frac{b}{s-1}$$

$$2 = a(s-1) + b(s+4)$$

$$\begin{cases} a+b=0 \\ -a+4b=2 \end{cases} \qquad b=\frac{2}{5},\ a=-\frac{2}{5}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = -\frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= -\frac{2}{5}e^{-4t} + \frac{2}{5}e^{t}$$

#9 $F(s) = \dfrac{1-2s}{s^2+4s+5}$

$$= \frac{1}{(s+2)^2+1} - \frac{2s}{(s+2)^2+1}$$

$$= \frac{1}{(s+2)^2+1} - 2\frac{s+2}{(s+2)^2+1} + \frac{4}{(s+2)^2+1}$$

$$= 5\left(\frac{1}{(s+2)^2+1}\right) - 2\left(\frac{s+2}{(s+2)^2+1}\right)$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$= 5\,\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} - 2\,\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\}$$

$$= 5e^{-2t}\sin t - 2e^{-2t}\cos t$$

$$= e^{-2t}(5\sin t - 2\cos t)$$

#13

$$y'' - 2y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1$$

$$s^2Y(s) - s\,y(0) - y'(0) - 2[sY(s) - y(0)] + 2Y(s) = 0$$

$$s^2Y(s) - 1 - 2sY(s) + 2Y(s) = 0$$

$$(s^2 - 2s + 2)Y(s) = 1$$

$$Y(s) = \frac{1}{(s-1)^2+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = e^{t}\sin t.$$

#15

$$y''-2y'+4y=0 \quad , \quad y(0)=2, \; y'(0)=0$$

$$s^2Y(s)-sy(0)-y'(0)-2[sY(s)-y(0)]+4Y(s)=0$$

$$s^2Y(s)-2s-2sY(s)+4+4Y(s)=0$$

$$(s^2-2s+4)Y(s)=2s-4$$

$$Y(s)=\frac{2s-4}{s^2-2s+1+3}=2\left(\frac{s-1}{(s-1)^2+3}\right)$$

$$+\frac{-2}{(s-1)^2+3}$$

$$=2\left(\frac{s-1}{(s-1)^2+3}\right)-\frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{(s-1)^2+3}\right)$$

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=2\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+3}\right\}-\frac{2}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{(s-1)^2+3}\right\}$$

$$=2e^t\cos\sqrt{3}t-\frac{2}{\sqrt{3}}e^t\sin\sqrt{3}t.$$

#23

$$y'' + 2y' + y = 4e^{-t} \qquad y(0) = 2, \quad y'(0) = -1$$

$$s^2 Y(s) - s y(0) - y'(0) + 2[sY(s) - y(0)] + Y(s) = 4\left(\frac{1}{s+1}\right)$$

$$s^2 Y(s) - 2s + 1 + 2sY(s) - 4 + Y(s) = \frac{4}{s+1}$$

$$(s^2 + 2s + 1)Y(s) - 2s - 3 = \frac{4}{s+1}$$

$$(s^2 + 2s + 1)Y(s) = \frac{4}{(s+1)} + (2s+3)$$

$$Y(s) = \frac{4}{(s+1)^3} + \frac{2s+3}{(s+1)^2}$$

$$= \frac{4}{(s+1)^3} + \frac{2(s+1)}{(s+1)^2} + \frac{1}{(s+1)^2}$$

$$= \frac{4}{(s+1)^3} + \frac{1}{(s+1)^2} + \frac{2}{s+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2\,\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^{2+1}}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + 2\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 2t^2e^{-t} + te^{-t} + 2e^{-t}$$