$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_1(t)g(t-1)\} = e^{-s}\,\mathcal{L}\{g(t)\} = e^{-s}(2/s^3 + 1/s)$.

14. Use partial fractions to write $F(s) = e^{-2s}\,\frac{1}{3}[\frac{1}{s-1} - \frac{1}{s+2}]$. For ease in calculations let us define $G(s) = (s-1)^{-1}$ and $H(s) = (s+2)^{-1}$. Then $F(s) = [e^{-2s}\,G(s) - e^{-2s}\,H(s)]/3$. Using the fact that $\mathcal{L}\{e^{at}\} = (s-a)^{-1}$ and applying Theorem 6.3.1, we have $F(s) = [e^{-2s}\,\mathcal{L}\{e^{t}\} - e^{-2s}\,\mathcal{L}\{e^{-2t}\}]/3$. Thus $F(s) = [\mathcal{L}\{u_2(t)e^{(t-2)}\} - \mathcal{L}\{u_2(t)e^{-2(t-2)}\}]/3$. Using the linearity of the Laplace transform, we have $\mathcal{L}\{f(t)\} = \mathcal{L}\{u_2(t)[e^{t-2} - e^{-2(t-2)}]/3\}$. Hence, $f(t) = [u_2(t)(e^{t-2} - e^{-2(t-2)})]/3$. An alternate method is to complete the square in the denominator: $F(s) = \dfrac{e^{-2s}}{(s+1/2)^2 - 9/4}$. From line 7, Table 6.2.1, this gives $f(t) = (2/3)u_2(t)e^{-(t-2)/2}\,\sinh\frac{3}{2}(t-2)$, which can be shown to be the same as that found above.

21. By completing the square in the denominator of F we can write $F(s) = (2s+1)/[(2s+1)^2 + 4]$. This has the form $G(2s+1)$ where $G(u) = u/(u^2+4)$. We must find $\mathcal{L}^{-1}\{G(2s+1)\}$. Applying the results of Prob. 19(c), with $a = 2$ and $b = 1$, we have $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}e^{-t/2}\cos(\frac{2t}{2})$, since $\mathcal{L}^{-1}\{G(s)\} = \cos 2t$.

22. If the approach of Prob. 21 is used we find $f(t) = (1/3)e^{2t/3}\sinh(t/3)$, which is equivalent to the given answer using the definition of sinht.

27. Assuming that term-by-term integration of the infinite series is permissible and recalling that $\mathcal{L}\{u_c(t)\} = e^{-cs}/s$ for $s > 0$, we have $\mathcal{L}\{f(t)\} = (1/s) + \sum_{k=1}^{\infty}(-1)^k\,\mathcal{L}\{u_k(t)\}$

$= (1/s) + \sum_{k=1}^{\infty}(-1)^k\,(e^{-ks}/s) = \frac{1}{s}\sum_{k=0}^{\infty}(-e^{-s})^k$. We recognize the last infinite series as the geometric series, $\sum_{k=0}^{\infty} ar^k$, with $a = 1$ and $r = -e^{-s}$. This series converges to $[1/(1+e^{-s})]$ if $|r| < 1$ (or $s > 0$). Hence, $\mathcal{L}\{f(t)\} = (1/s)[1/(1+e^{-s})]$, $s > 0$.

28. Using the definition of the Laplace transform we have $F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt$. Since f is periodic with period T, we have $f(t+T) = f(t)$. This suggests that we rewrite the improper integral as $\int_0^{\infty} e^{-st} f(t)dt = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)dt$. The periodicity of f also suggests that we make the change of variable $t = r + nT$. Hence,

$$F(s) = \sum_{n=0}^{\infty} \int_0^T e^{-s(r+nT)} f(r+nT)dr = \sum_{n=0}^{\infty} (e^{-sT})^n \int_0^T e^{-rs} f(r)dr,$$

where we have used the fact that $f(r+nT) = f(r+(n-1)T) = \ldots = f(r+T) = f(r)$, since f is periodic. We recognize this last series as the geometric series, $\sum_{n=0}^{\infty} au^n$, with $a = \int_0^T e^{-rs} f(r)dr$ and $u = e^{-sT}$. The geometric series converges to $a/(1-u)$ for $|u| < 1$ and consequently we obtain

$$F(s) = (1 - e^{-sT})^{-1} \int_0^T e^{-rs} f(r)dr, \quad s > 0.$$

30. The function f is periodic with period 2. The result of Prob. 28 gives us $\mathcal{L}\{f(t)\} = \int_0^2 e^{-st} f(t)dt/(1-e^{-2s})$. Calculating the integral we have

$$\begin{aligned} \int_0^2 e^{-st} f(t)dt &= \int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt \\ &= (1-e^{-s})/s + (e^{-2s}-e^{-s})/s \\ &= (e^{-2s}-2e^{-s}+1)/s \\ &= (1-e^{-s})^2/s. \end{aligned}$$

Since the denominator of $\mathcal{L}\{f(t)\}$, $1 - e^{-2s}$, may be written as $(1-e^{-s})(1+e^{-s})$ we obtain the desired answer.

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1. $f(t)$ can be written in the form $f(t) = 1 - u_{\pi/2}(t)$ and thus the Laplace transform of the D.E. is $(s^2+1)Y(s) - sy(0) - y'(0) = (1/s) - e^{-\pi s/2}/s$. Using the I.C. and solving for $Y(s)$, we obtain $Y(s) = (s^2+1)^{-1} + [s(s^2+1)]^{-1} - e^{-\pi s/2}/s(s^2+1)$. Using partial fractions on the second and third terms we find $Y(s)=(s^2+1)^{-1} + (1/s) - s/(s^2+1) - e^{-\pi s/2}/s + e^{-\pi s/2}s/(s^2+1)$. The inverse transform of the first three terms can be obtained directly from Table 6.2.1. Using Theorem 6.3.1 to find the inverse transform of the last two terms we have $\pounds^{-1}\{e^{-\pi s/2}/s\} = u_{\pi/2}(t)g(t - \pi/2)$ where

$g(t) = \pounds^{-1}\{1/s\} = 1$ and

$\pounds^{-1}\{e^{-\pi s/2}s/(s^2+1)\} = u_{\pi/2}(t)h(t - \pi/2)$ where

$h(t) = \pounds^{-1}\{s/(s^2+1)\} = \cos t$. Hence,

$y = 1 + \sin t - \cos t + u_{\pi/2}(t)[\cos(t - \pi/2) - 1]$

$= 1 + \sin t - \cos t - u_{\pi/2}(t)[1 - \sin t]$. The graph of the forcing function is a unit pulse for $0 \le t < \pi/2$ and 0 thereafter. The graph of the solution will be composed of two segments. The first, for $0 \le t < \pi/2$, is a sinusoid oscillating about 1, which represents the system response to a unit forcing function and the given initial conditions. For $t \ge \pi/2$, the forcing function, $f(t)$, is zero and the "initial" conditions are

$y(\pi/2) = \lim_{t\to\pi^-/2}(1 + \sin t - \cos t) = 2$ and

$y'(\pi/2) = \lim_{t\to\pi^-/2}(\cos t + \sin t) = 1$. For $t \ge \pi/2$ the system response is $y(t) = 2\sin t - \cos t$, which is a sinusoid oscillating about zero.

3. According to Theorem 6.3.1, $\pounds\{u_{2\pi}(t)\sin(t-2\pi)\} = e^{-2\pi s}\,\pounds\{\sin t\} = e^{-2\pi s}/(s^2+1)$. Transforming the D.E., we have $(s^2+4)Y(s) - sy(0) - y'(0) = 1/(s^2+1) - e^{-2\pi s}/(s^2+1)$. Using the I.C. and solving for $Y(s)$, we obtain $Y(s) = (1-e^{-2\pi s})/(s^2+1)(s^2+4)$. We apply partial fractions to write $Y(s) = [s^2+1)^{-1} - (s^2+4)^{-1} - e^{-2\pi s}(s^2+1)^{-1} + e^{-2\pi s}(s^2+4)^{-1}]/3$. We compute the inverse transform of the first two terms directly from Table 6.2.1 after noting that $(s^2+4)^{-1} = (1/2)[2/(s^2+4)]$. We apply Theorem 6.3.1 to the last two terms to obtain the solution,

$y = (1/3)\{\sin t - (1/2)\sin 2t - u_{2\pi}(t)[\sin(t-2\pi) - (1/2)\sin 2(t-2\pi)]\}$.
This may be simplified, using trigonometric identities, to $y = [(2\sin t - \sin 2t)(1-u_{2\pi}(t))]/6$. Note that the forcing function is $\sin t - \sin(t-2\pi) = 0$ for $t \geq 2\pi$. The solution is $y(t) = 2\sin t - \sin 2t$ for $0 \leq t < 2\pi$. Thus $y(2\pi^-) = 0$ and $y'(2\pi^-) = 2\cos 2\pi - 2\cos 4\pi = 0$. Hence the "initial" value problem for $t \geq 2\pi$ is $y'' + 4y = 0$, $y(2\pi) = 0$, $y'(2\pi) = 0$, which has the trivial solution $y = 0$. [Note that $1 - u_{2\pi}(t) = 0$ for $t \geq 2\pi$ so this agrees with the solution y given above].

8. Taking the Laplace transform, applying the I.C. and using Theorem 6.3.1 we have $(s^2+s+5/4)Y(s) = (1-e^{-\pi s/2})/s^2$. Thus

$$Y(s) = \frac{1-e^{-\pi s/2}}{s^2(s^2+s+5/4)}$$
$$= (1-e^{-\pi s/2})\left\{\frac{4/5}{s^2} - \frac{16/25}{s} + \frac{(16/25)s-4/25}{(s+1/2)^2+1}\right\}$$

$= (1-e^{-\pi s/2})H(s)$, where we have used partial fractions and completed the square in the denominator of the last term. Since the numerator of the last term of H can be written as $\frac{16}{25}[(s+1/2) - 3/4]$, we see that

$\mathcal{L}^{-1}\{H(s)\} = (4/25)(5t - 4 + 4e^{-t/2}\cos t - 3e^{-t/2}\sin t)$, which yields the desired solution. The graph of the forcing function is a ramp ($f(t) = t$) for $0 \leq t < \pi/2$ and a constant ($f(t) = \pi/2$) for $t \geq \pi/2$. The solution will be a damped sinusoid oscillating about the "ramp" $(20t-16)/25$ for $0 \leq t < \pi/2$ and oscillating about $2\pi/5$ for $t \geq \pi/2$.

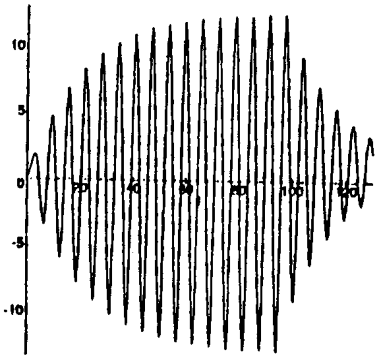
10. Note that $g(t) = \sin t - u_\pi(t)\sin t = \sin t + u_\pi(t)\sin(t-\pi)$. Proceeding as in Prob. 8 we find

$Y(s) = (1+e^{-\pi s})\dfrac{1}{(s^2+1)(s^2+s+5/4)}$. The correct partial fraction expansion of the quotient is $\dfrac{as+b}{s^2+1} + \dfrac{cs+d}{s^2+s+5/4}$, where

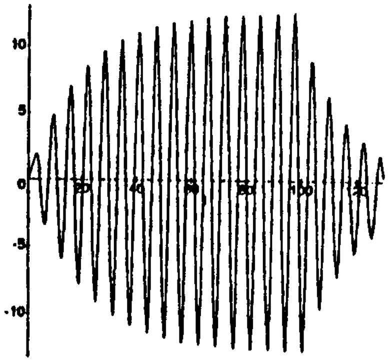
$a+c = 0$, $a+b+d = 0$, $(5/4)a+b+c = 0$ and $(5/4)b+d = 1$ by equating coefficients. Solving for a,b,c,d and following the steps of Prob. 8 yields the desired solution.

16b. Taking the Laplace transform of the D.E. we obtain $U(s^2 + s/4 + 1) = k(e^{-3s/2}-e^{-5s/2})/s$, since the I.C. are zero. Solving for U and using partial fractions yields

20a. $y(t)$ for $n = 30$

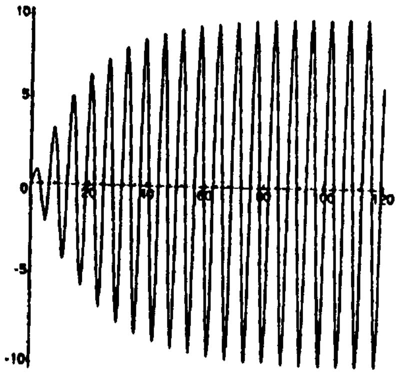


$y(t)$ for $n = 31$



20b. From the graph of part a, $A \cong 12.5$ and the frequency is 2π.

20c. From the graph (or analytically) $A = 10$ and the frequency is 2π.



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1. Proceeding as in Example 1, we take the Laplace transform of the D.E. and apply the I.C.: $(s^2 + 2s + 2)Y(s) = s + 2 + e^{-\pi s}$. Thus, $Y(s) = (s+2)/[(s+1)^2 + 1] + e^{-\pi s}/[(s+1)^2 + 1]$. We write the first term as $(s+1)/[(s+1)^2 + 1] + 1/[(s+1)^2 + 1]$. Applying Theorem 6.3.1 and using Table 6.2.1, we obtain the solution, $y = e^{-t}\cos t + e^{-t}\sin t + u_\pi(t)e^{-(t-\pi)}\sin(t-\pi)$.

3. Taking the Laplace transform and using the I.C. we have $(s^2+3s+2)Y(s) = \frac{1}{2} + e^{-5s} + \frac{e^{-10s}}{s}$. Thus $Y(s) = \frac{1/2}{s^2+3s+2} + \frac{e^{-5s}}{s^2+3s+2} + e^{-10s}(\frac{1/2}{s} + \frac{1/2}{s+2} - \frac{1}{s+1})$ and hence $y(t) = \frac{1}{2}h(t) + u_5(t)h(t-5) + u_{10}(t)[\frac{1}{2}+\frac{1}{2}e^{-2(t-10)}-e^{-(t-10)}]$ where $h(t) = e^{-t} - e^{-2t}$.

5. The Laplace transform of the D.E. is $(s^2+2s+3)Y(s) = \frac{1}{s^2+1} + e^{-3\pi s}$, so $Y(s) = \frac{1}{(s^2+1)(s^2+2s+3)} + e^{-3\pi s}[\frac{1}{s^2+2s+3}]$. Using partial fractions or a computer algebra system we obtain $y(t) = \frac{1}{4}\sin t - \frac{1}{4}\cos t + \frac{1}{4}e^{-t}\cos\sqrt{2}\,t + \frac{1}{\sqrt{2}}u_{3\pi}(t)h(t-3\pi)$, where $h(t) = e^{-t}\sin\sqrt{2}\,t$.

7. Taking the Laplace transform of the D.E. yields $(s^2+1)Y(s) - y'(0) = \int_0^\infty e^{-st}\delta(t-2\pi)\cos t\,dt$. Since $\delta(t-2\pi) = 0$ for $t \neq 2\pi$ the integral on the right is equal to $\int_{-\infty}^\infty e^{-st}\delta(t-2\pi)\cos t\,dt$ which equals $e^{-2\pi s}\cos 2\pi$ from Eq.(16). Substituting for $y'(0)$ and solving for $Y(s)$ gives $Y(s) = \frac{1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1}$ and hence

$$y(t) = \sin t + u_{2\pi}(t)\sin(t-2\pi) = \begin{cases} \sin t & 0 \le t < 2\pi \\ 2\sin t & 2\pi \le t \end{cases}.$$

10. Follow the same steps as in the solution for Prob. 7.

13a. From Eq. (22) $y(t)$ will complete one cycle when $\sqrt{15}(t-5)/4 = 2\pi$ or $T = t - 5 = 8\pi/\sqrt{15}$, which is consistent with the plot in Fig. 6.5.3. Since an impulse causes a discontinuity in the first derivative, we need to find the value of y' at $t = 5$ and $t = 5 + T$. From Eq. (22) we have, for $t \ge 5$, $y' = e^{-(t-5)/4}[\frac{-1}{2\sqrt{15}}\sin\frac{\sqrt{15}}{4}(t-5) + \frac{1}{2}\cos\frac{\sqrt{15}}{4}(t-5)]$. Thus $y'(5) = \frac{1}{2}$ and $y'(5+T) = \frac{1}{2}e^{-T/4}$. Since the original impulse, $\delta(t-5)$, caused a discontinuity in y' of 1/2 at $t = 5$, we must choose the impulse at $t = 5 + T$ to be $-e^{-T/4}$, which is equal and opposite to y' at $5 + T$.

13b. Now consider $2y'' + y' + 2y = \delta(t-5) + k\delta(t-5-T)$ with $y(0) = 0$, $y'(0) = 0$. Using the results of Ex. 1 we have