§2.3 #6

(a)

$$\begin{cases} \frac{dv}{dt} = g \\ v(0) = 0 \end{cases} \Rightarrow v(t) = gt \qquad \begin{cases} \frac{ds}{dt} = v = gt \\ s(0) = 0 \end{cases} \Rightarrow s(t) = \frac{1}{2}gt^2$$

when $s(t) = h \Rightarrow \frac{1}{2}gt^2 = h$, $t = \sqrt{2h/g}$, and $v(t) = g \cdot \sqrt{2h/g} = \sqrt{2gh}$

(b) The volume decreases in the tank is in the rate $\left(\text{Volume}/\text{time}\right)$ of $A(h)\frac{dh}{dt}$ (note: $dh/dt < 0$)

This rate is equivalent to that flowing out from the outlet, for which is $-\alpha a v(t) = -\alpha a \sqrt{2gh}$

Hence, when equating the rate of change, we have

$$A(h)\frac{dh}{dt} = -\alpha a \sqrt{2gh}.$$

(c)

$$A(h) = \pi r^2 = \pi \cdot 1^2 = \pi$$

$g = 9.8\ m/sec^2$, $\alpha = 0.6$, $a = \pi r^2 = \pi(0.1)^2 = 0.01\pi$

Hence the initival value problem is

$$\frac{dh}{dt} = -0.006\sqrt{19.6h}, \quad h(0) = 3.$$

$$\frac{d\,h(t)}{\sqrt{h}} = -0.0266\,dt, \quad 2h^{\frac{1}{2}} = -0.0266t + C$$

$h(0) = 3 \Rightarrow C = 2\cdot\sqrt{3} = 3.4641$, Hence, when $h = 0$, $t = \frac{3.4641}{0.0266} \sim 130$.

#7

(a)
$$\begin{cases} \dfrac{dS}{dt} = rS \\ S(0) = S_0 \end{cases}$$

$$S(t) = S_0 e^{rt}$$

$S(t) = 2S_0$. implies $e^{rt} = 2$. $rt = \ln 2$

$\Rightarrow T = \ln 2 / r$ years

(b)

$r = 7\%$. $T = \ln(2) / 0.07 = 9.90$ years

(c)

$$S_0 e^{r8} = 2S_0$$

$$e^{r8} = 2$$

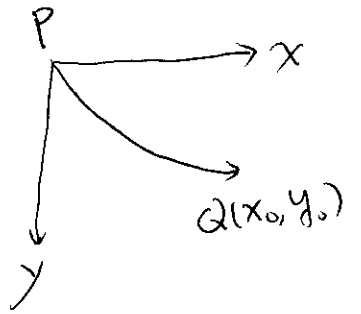
$$r8 = \ln(2)$$

$$r = \ln(2)/8 = 0.0866 = 8.66\%$$

#32 $\quad (1+y'^2)y = k^2$

(a) $\quad y'^2 = \frac{k^2}{y} - 1$

$$y' = \pm\sqrt{\frac{k^2}{y} - 1} \Rightarrow y' = \sqrt{\frac{k^2 - y}{y}}$$



Because of the gravity, we require $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} > 0$, for which $\Delta y > 0$ when $\Delta x > 0$

(b) $\quad y = k^2 \sin^2 t$

$$y' = \sqrt{\frac{k^2(1-\sin^2 t)}{k^2 \sin^2 t}} = \frac{\cos t}{\sin t} = \cot(t)$$

$$2k^2 \sin^2 t \, dt = dx$$

$$x = 2k^2\left(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t + c_1\right), \quad t=0,\ x=0, \quad c_1 = 0$$

$$dy = \cot(t)\, dx = \cot(t)\, 2k^2 \sin^2 t \, dt$$

$$dy = 2k^2 \sin t \cos t \, dt = k^2 \sin 2t \, dt$$

$$y = \frac{-1}{2}k^2(\cos 2t + c_2), \quad \text{since } t=0,\ y=0. \Rightarrow c_2 = -1$$

(c) $\quad \theta = 2t.$

$$\begin{cases} y = -\frac{1}{2}k^2(\cos\theta - 1) = k^2(1-\cos\theta)/2 \\ x = \frac{1}{2}k^2\theta - \frac{1}{2}k^2 \sin\theta = k^2(\theta - \sin\theta)/2 \end{cases}$$

(d)

$$1 = k^2(\theta - \sin\theta)/2 \quad -①$$

$$2 = k^2(1 - \cos\theta)/2 \quad -②$$

$\frac{①}{②} \Rightarrow$ Solve $\frac{1}{2} = \frac{\theta - \sin\theta}{1 - \cos\theta}$ for $\theta = 1.401379436$;

after substitution k is found to be 2.19344615.

$\theta = 1.401379436 \rightarrow ①$ or ②

$\Rightarrow k = 2.19344615$.

§2.4

#4

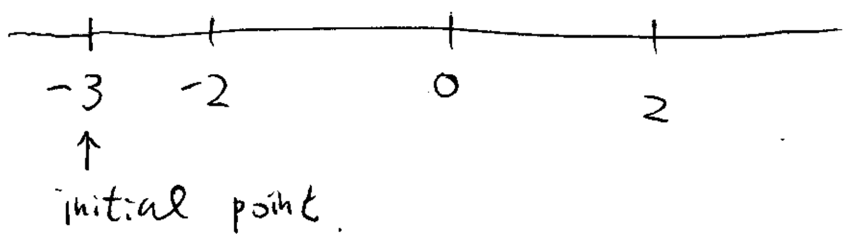
$$(4-t^2)y' + 2ty = 3t^2 \quad , \quad y(-3) = 1$$

$$y' + \left(\frac{2t}{4-t^2}\right)y = \frac{3t^2}{4-t^2}$$

$$p(t) = \frac{2t}{4-t^2}$$

$$g(t) = \frac{3t^2}{4-t^2}$$

$p(t)$ and $g(t)$ are continuous when $t \neq \pm 2$



Hence, the interval is $(-\infty, -2)$ or $-\infty < t < -2$

#11

$$\frac{dy}{dt} = \frac{1+t^2}{3y-y^2}$$

$$f(t,y) = \frac{1+t^2}{3y-y^2}$$

$f(t,y)$ is discontinuous at $3y - y^2 = 0$, $y = 0$ and $y = 3$

$$\frac{\partial f}{\partial y} = -(1+t^2)(3y-y^2)^{-2}(3-2y)$$

$\frac{\partial f}{\partial y}$ is discontinuous at $y=0$ and $y=3$

Hence, on the t-y plane. Theorem 2.4.2

are satisfied if $y \neq 0$ and $y \neq 3$.

#16 (#4)

$$y' = t^2 / y(1+t^3) \qquad y(0) = y_0$$

$$yy' = t^2 / 1+t^3$$

$$\frac{d}{dt}\left(\frac{1}{2}y^2\right) = \frac{t^2}{1+t^3}$$

$$\int_0^t \frac{d}{ds}\left(\frac{1}{2}y^2(s)\right) = \int_0^t \frac{t^2}{1+t^3}$$

LHS : $\frac{1}{2}y^2(s)\Big|_0^t = \frac{1}{2}y^2(t) - \frac{1}{2}y_0^2$

RHS : Let $1+s^3 = z$, $3s^2 ds = dz$, $s^2 ds = \frac{1}{3}z$

$$\int \frac{dz}{3z} = \frac{1}{3}\ln(1+s^3)\Big|_0^t = \ln(1+t^3)^{1/3}$$

Hence $\frac{1}{2}y^2(t) = \ln(1+t^3)^{1/3} + \frac{1}{2}y_0^2$

$$y(t) = \pm\sqrt{2\ln(1+t^3)^{1/3} + y_0^2}$$

$$= \pm\sqrt{2/3\,\ln(1+t^3) + y_0^2}$$

we require

$$\frac{2}{3}\ln(1+t^3)+y_0^2>0$$

or $$\ln(1+t^3)>-\frac{3}{2}y_0^2$$

$$1+t^3> e^{-\frac{3}{2}y_0^2}$$

$$t^3> e^{-\frac{3}{2}y_0^2}-1$$

$$t>\left(e^{-\frac{3}{2}y_0^2}-1\right)^{1/3} \quad \left(\text{or } t>-(1-e^{-\frac{3}{2}y_0^2})^{1/3}\right.$$

$$(e^{-\frac{3}{2}y_0^2}-1)^{1/3}<t<\infty$$

#22
$$y' = \frac{-t+(t^2+4y)^{1/2}}{2} \qquad y(2)=-1$$

(a) $y_1(t)=1-t$,

I.C: $y(2) = 1-2=-1$ is satisfied

Plug in $y_1(t)$ in the above equation

$LHS = y' = -1$

$$RHS = \frac{-t+(t^2+4(1-t))^{1/2}}{2}$$

$$= \frac{-t \pm \sqrt{(t-2)^2}}{2}$$

In order to cancel t, to obtain $RHS=-1$,

we require $(t-2)>0$, for which $\sqrt{(t-2)^2} = t-2$.

Hence $y_1(t) = 1-t$ is a slout:on for $t>2$.

Similarly
$$y_2(t) = -t^2/2$$

y_2 satisfies the initial condition and upon substituting

y_2 into the ODE, we have $y_2' = -\frac{t}{2} = f(t, y_2)$.

$\Rightarrow$ $y_2(t)$ is a solution for all t.

(b) $f(t,y)=\dfrac{-t+(t^2+4y)^{1/2}}{2}$ is continuous for all t and y

$\Rightarrow$ solutions exist.

$$\frac{\partial f}{\partial y} = \frac{1}{4}(t^2+4y)^{-\frac{1}{2}}\cdot 4 = (t^2+4y)^{-\frac{1}{2}}$$

If $t^2+4y=0$, or $(t,y)=\left(t,-\frac{t^2}{4}\right)$

the function $\frac{\partial f}{\partial y}$ is not continuous. In particular, when $t=2$, $y=-1$, where is the initial condition.

Therefore, there are no interval that contains $y(2)=-1$ and has unique solution.

(c) If $y=ct+c^2$, c is an arbitrary constant

we have

$$y'=c=f(t,y), \quad \text{for } t>-2c.$$

In particular, when $c=-1$, $y=y_1(t)=1-t$, and $y_1(2)=-1$.

Since $y_2(t)$ is a polynomial of degree 2, $y_2(t)=-t^2/4$, which cannot be represented by $y=ct+c^2$ which is a family of polynomial of degree 1.