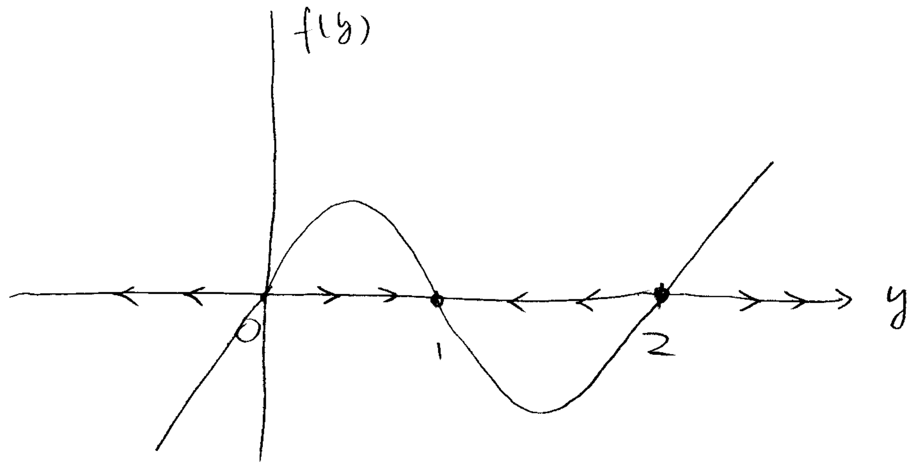
§ 2.5

#3 $\dfrac{dy}{dt} = y(y-1)(y-2)$ $\quad y_0 > 0$

critical points $y=0$, $y=1$, and $y=2$



$\left.\begin{array}{l} y=0 \\ y=2 \end{array}\right\}$ unstable ; $y=1$ asymptotically stable

$$\frac{d^2y}{dt^2} = f'(y)\,f(y) = (3y^2-6y+2)\,y(y-1)(y-2)$$

$f'(y)=0$ gives the inflection points

$$y = \frac{+6 \pm \sqrt{36-24}}{6} = \frac{6 \pm \sqrt{12}}{6} = \frac{3 \pm \sqrt{3}}{3}$$

$y'' = \dfrac{d^2y}{dt^2} > 0 \quad y > 2$, concave up

$y'' > 0 \quad \dfrac{3+\sqrt{3}}{3} < y < 2$, concave down

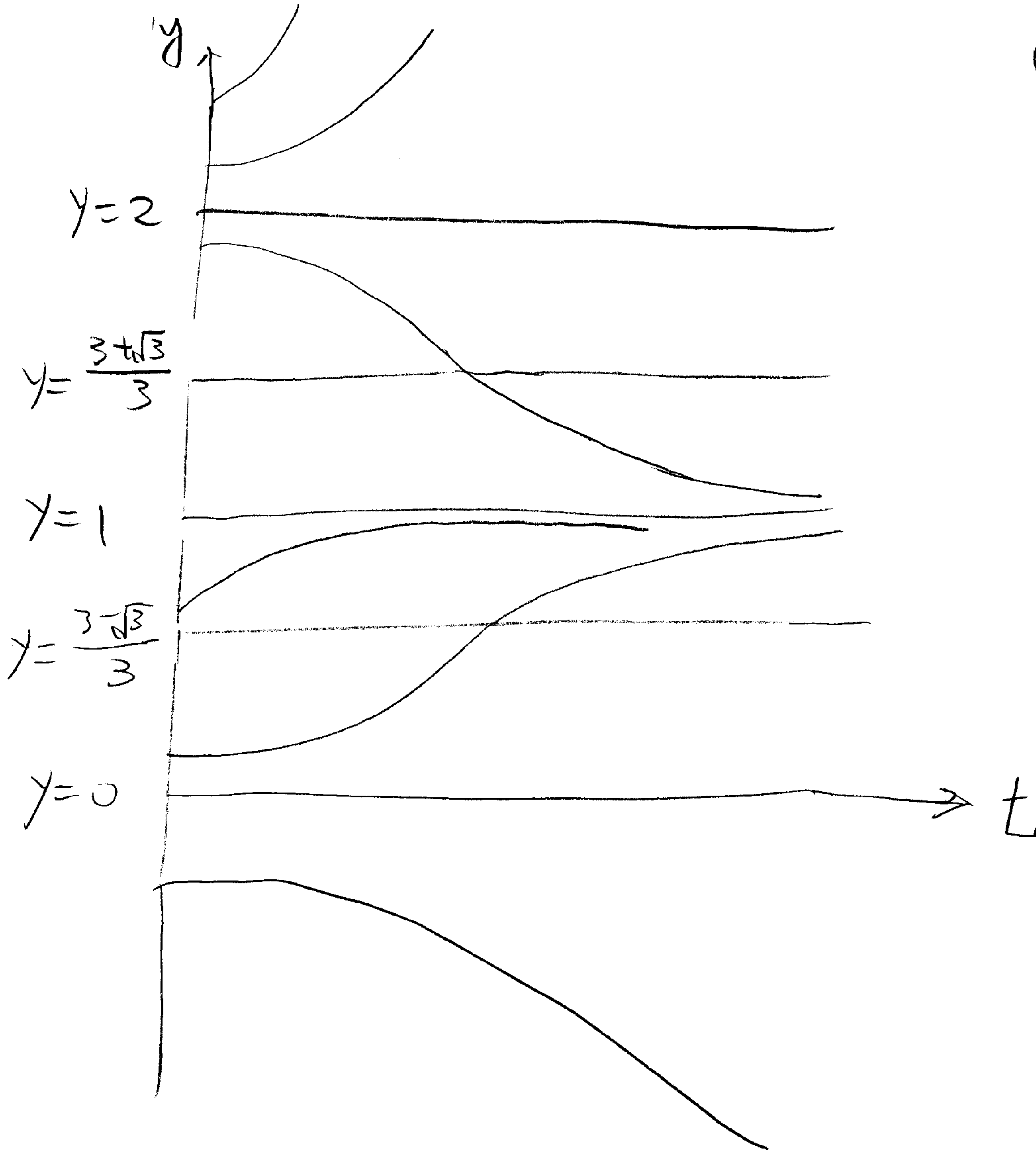
$y'' > 0 \quad 1 < y < \dfrac{3+\sqrt{3}}{3}$ concave up

$y'' < 0 \quad \dfrac{3-\sqrt{3}}{3} < y < 1$ concave down

$y'' > 0 \quad 0 < y < \dfrac{3-\sqrt{3}}{3}$ concave up

$y'' < 0 \quad y < 0$ concave down

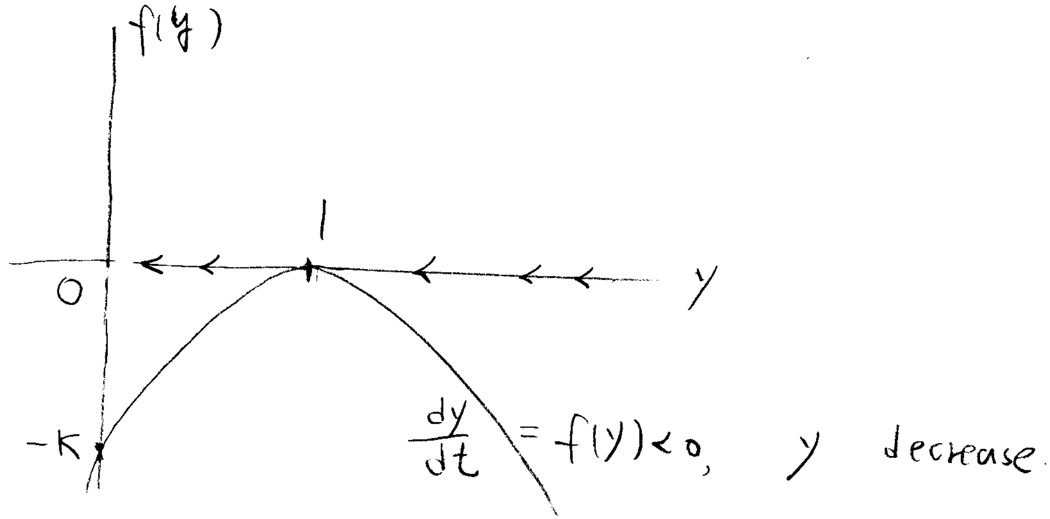
②

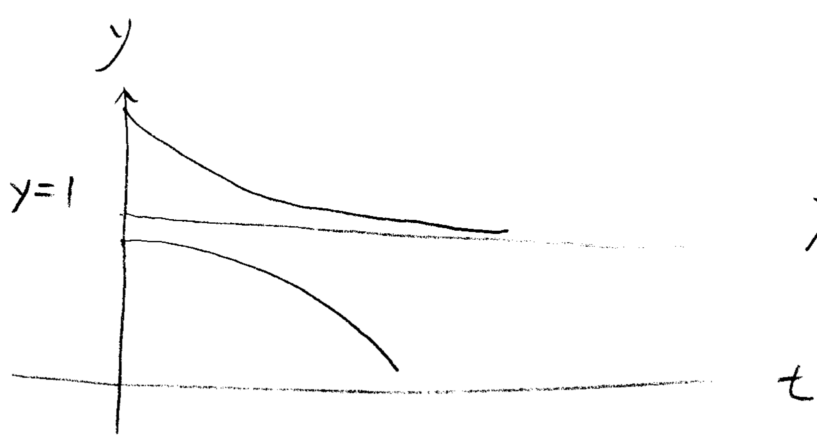


8

$$\frac{dy}{dt} = -k(y-1)^2 \qquad k>0, \quad -\infty < y_0 < \infty$$

equilibrium point $y=1$ $\left(f(y) = -k(y-1)^2 = 0 \text{ at } y=1\right)$





$y=1$ is semistable

$$\frac{d^2y}{dt^2} = f'(y)\,f(y) = \left[-2k(y-1)\right]\left[-k(y-1)^2\right] = 2k^2(y-1)^3$$

$y<1$, $\frac{d^2y}{dt^2} < 0$ concave down

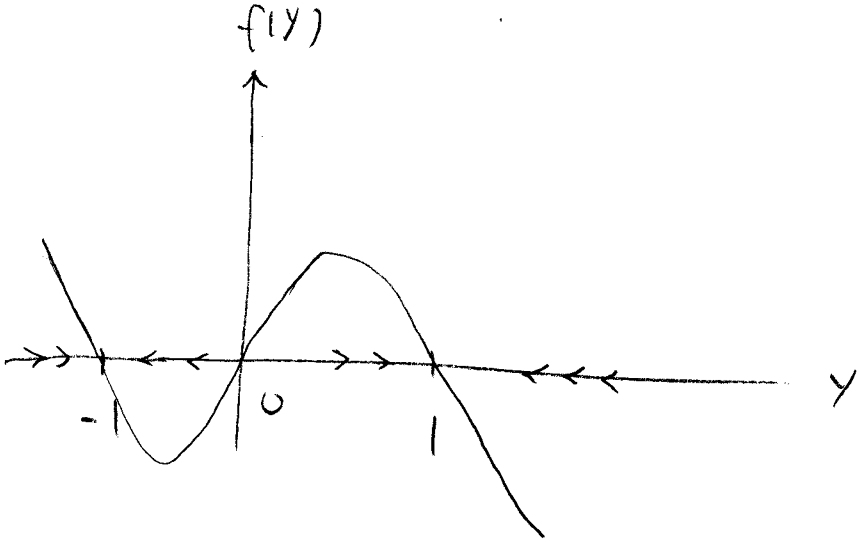
$y>1$, $\frac{d^2y}{dt^2} > 0$ concave up

#10

$$\frac{dy}{dt} = y(1-y^2) \qquad -\infty < y_0 < \infty$$

$f(y) = y(1-y^2)$

critical points $y=0$ $y=1,$ $y=-1$



$y=0$ unstable equilibrium

$y=\pm 1$ asymptotically stable

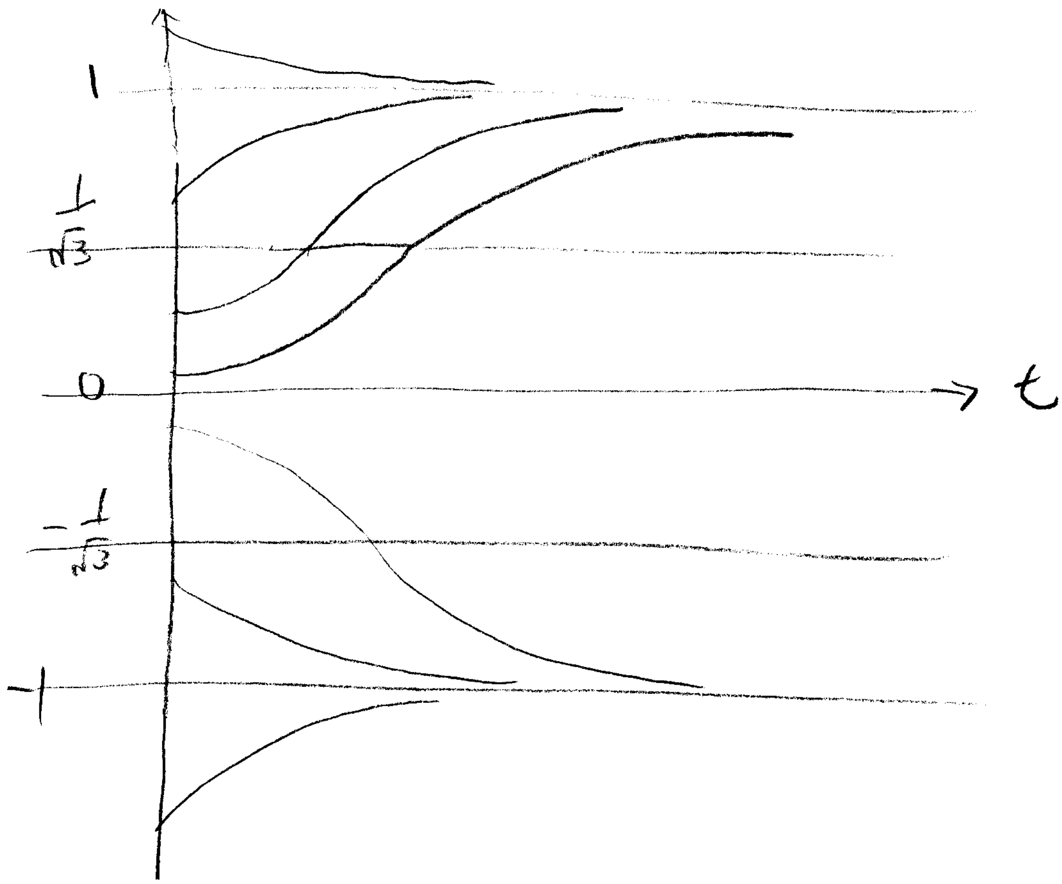
$$\frac{d^2y}{dt^2} = f'(y)\,f(y) = (1-3y^2)(y-y^3)$$

$\frac{d^2y}{dt^2} < 0$ $y<-1$ c.d $\qquad \frac{d^2y}{dt^2} > 0,$ $0<y<\frac{1}{\sqrt{3}}$ c.u

$\frac{d^2y}{dt^2} > 0$ $-1<y<\frac{-1}{\sqrt{3}}$ c.u $\qquad \frac{d^2y}{dt^2} < 0,$ $\frac{1}{\sqrt{3}}<y<1$ c.d

$\frac{d^2y}{dt^2} < 0$ $\frac{-1}{\sqrt{3}}<y<0$ c.d $\qquad \frac{d^2y}{dt^2} > 0$ $y>1$ c.u

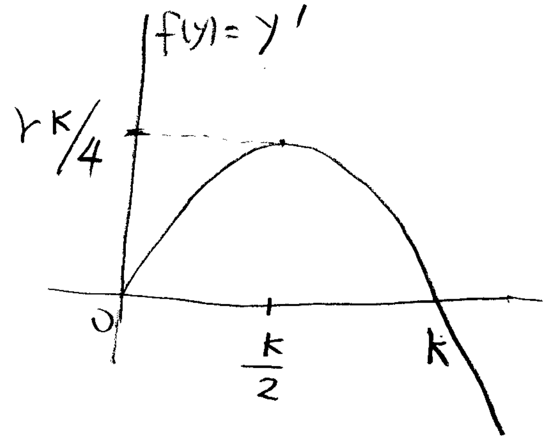
c.u: concave up; c.d: concave down.

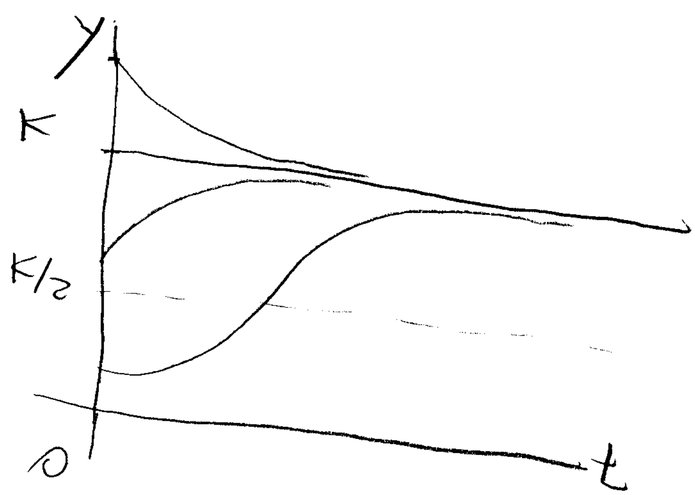


#15

$$\frac{dy}{dt} = ry\left[1-(y/K)\right]$$

logistic growth





$$y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

(a) $y_0 = K/3$

$$y(t) = \frac{K^2/3}{K/3 + (K - \frac{K}{3})e^{-rt}}$$

$$= \frac{K^2}{K + 2Ke^{-rt}}$$

$$= \left(\frac{3}{1+2e^{-rt}}\right) K/3,$$ when $y(t) = 2y_0 = 2 \cdot \left(\frac{K}{3}\right)$

we have

$$\frac{3}{1+2e^{-rt}} = 2 \Rightarrow 3 = 2 + 4e^{-rt}$$

$$e^{-rt} = \frac{1}{4}$$

$$-rt = \ln 2^{-2} = -2\ln(2)$$

$$\boxed{\tau = \frac{2\ln(2)}{r}}$$

when $r = 0.025 = \frac{1}{40}$

$$\tau = 2\ln(2) \cdot 40 = 80\ln(2) = 55.452$$

(b)

$$y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

If $y_0 = \alpha K$

$$y(t) = \frac{\alpha K^2}{\alpha K + (1-\alpha)K e^{-rt}}$$

$$= \frac{\alpha K}{\alpha + (1-\alpha)e^{-rt}} = \left(\frac{\alpha}{\alpha + (1-\alpha)e^{-rt}}\right)K$$

$$= \beta K$$

$$\Rightarrow \quad \beta = \frac{\alpha}{\alpha + (1-\alpha)e^{-rt}}$$

$$\alpha\beta + (1-\alpha)\beta e^{-rt} = \alpha$$

$$(1-\alpha)\beta e^{-rt} = \alpha(1-\beta)$$

$$e^{-rt} = \frac{\alpha(1-\beta)}{\beta(1-\alpha)}$$

$$T = \ln\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right) \cdot \frac{-1}{r} = \ln\left[\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^{-\frac{1}{r}}\right]$$

$$= \ln\left[\left(\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right)^{\frac{1}{r}}\right] = \frac{1}{r}\ln\left(\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right)$$

Since $r > 0$, when $\alpha = 0$ $\beta = 1$, the denominato $= 0$

$T \to \infty$

$$T = \ln\left[\left(\frac{(0.9)(0.9)}{(0.1)\cdot(0.1)}\right)\right]\frac{1}{0.025}$$

$$= \ln(3^4)40 = 160\ln(3) = 175.78$$

#25 $\frac{dy}{dt} = a - y^2$

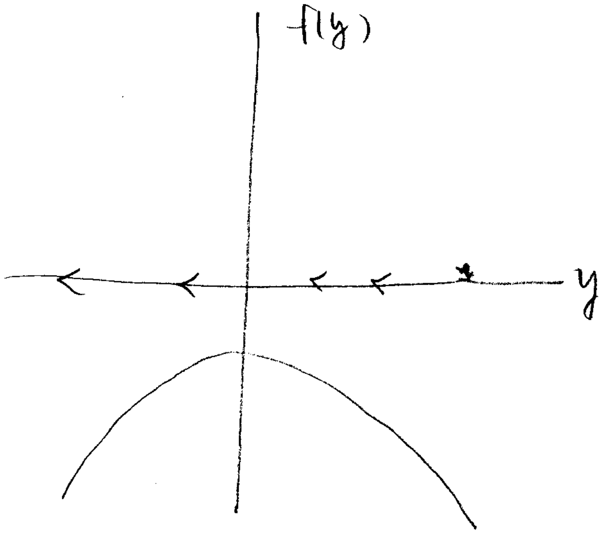
$f(y) = a - y^2 = 0 \Rightarrow a = y^2$

(a) If $a < 0$, y is <u>not</u> a real number, hence no critical points

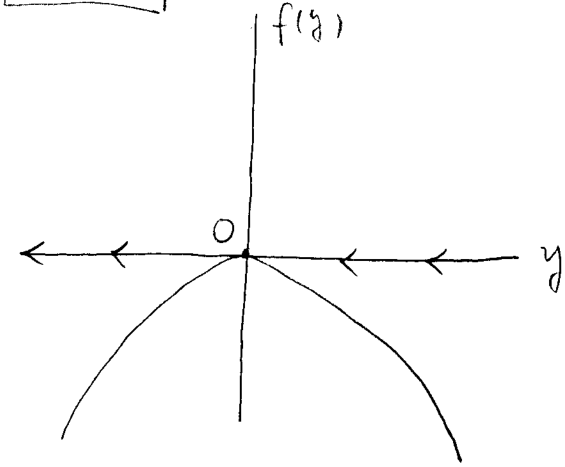
$a = 0$, $y = 0$ is the only critical point

$a > 0$ $y = \pm\sqrt{a}$ are the critical points

(b) $\boxed{\text{case I}}$ $a<0$ (no critical point

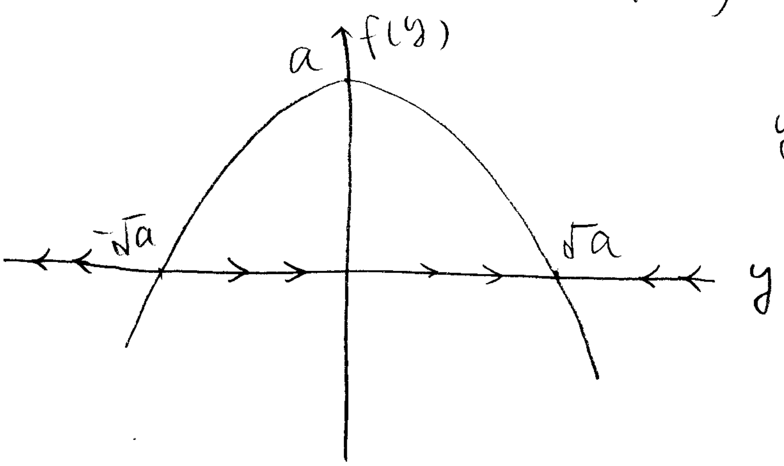


$\boxed{\text{case II}}$ $a=0$, $(y, f(y)) = (0,0)$ is a critical point



$y=0$ is a semistable critical point.

$\boxed{\text{case III}}$ $a>0$ $(y, f(y)) = (\pm\sqrt{a}, 0)$ are critical points



$y=\sqrt{a}$ is asymptotically stable

$y=-\sqrt{a}$ is unstable.

(c) $\boxed{\text{case I}}$ $a < 0$

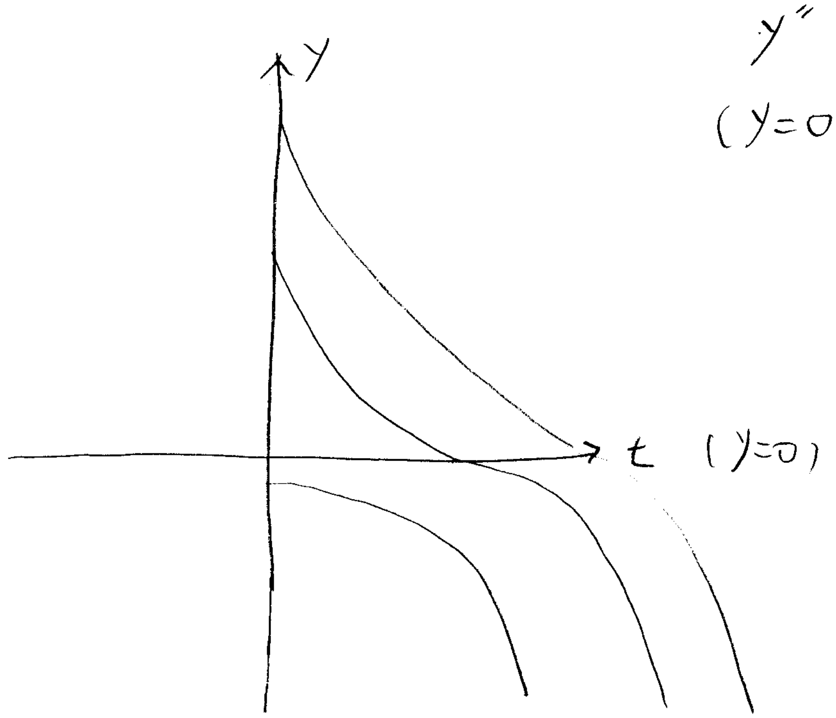
$$y'' = \frac{d^2y}{dt^2} = f'(y)f(y) = (-2y)(a-y^2)$$

since $a<0$, $f(y) = a - y^2 < 0$, $y'' > 0$, if $y > 0$ (concave up)

$y'' < 0$ if $y < 0$ (concave down)

$y'' = 0$ if $y = 0$

($y=0$ is an inflection point)



$\boxed{\text{case II}}$ $a = 0$

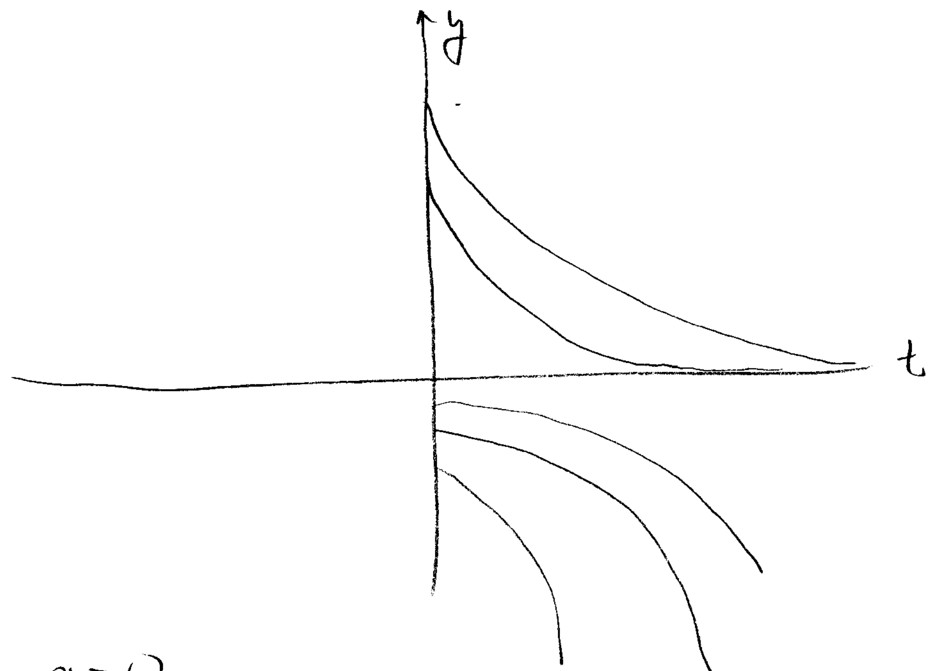
$$y'' = \frac{d^2y}{dt^2} = (-2y)(-y^2) = 2y^3$$

$y'' = 0$ when $y=0$; $y=0$ is an inflection point

$y'' > 0$ when $y>0$; $y''>0$ means concave up

$y'' < 0$ 〃 $y<0$; $y''<0$ 〃 〃 down

$y=0$ is a semistable critical point



$\boxed{\text{Case III}}$ $a > 0$

$$y'' = \frac{d^2y}{dt^2} = (-2y)(a - y^2)$$

$y = 0$ is the inflection point. $(f'(y) = 0)$

$y = +\sqrt{a}$ is asymptotically stable

$y = -\sqrt{a}$ is unstable

$y'' > 0$ if $y > +\sqrt{a}$

$y'' < 0$ if $y < -\sqrt{a}$

$y'' < 0$ if $0 < y < +\sqrt{a}$

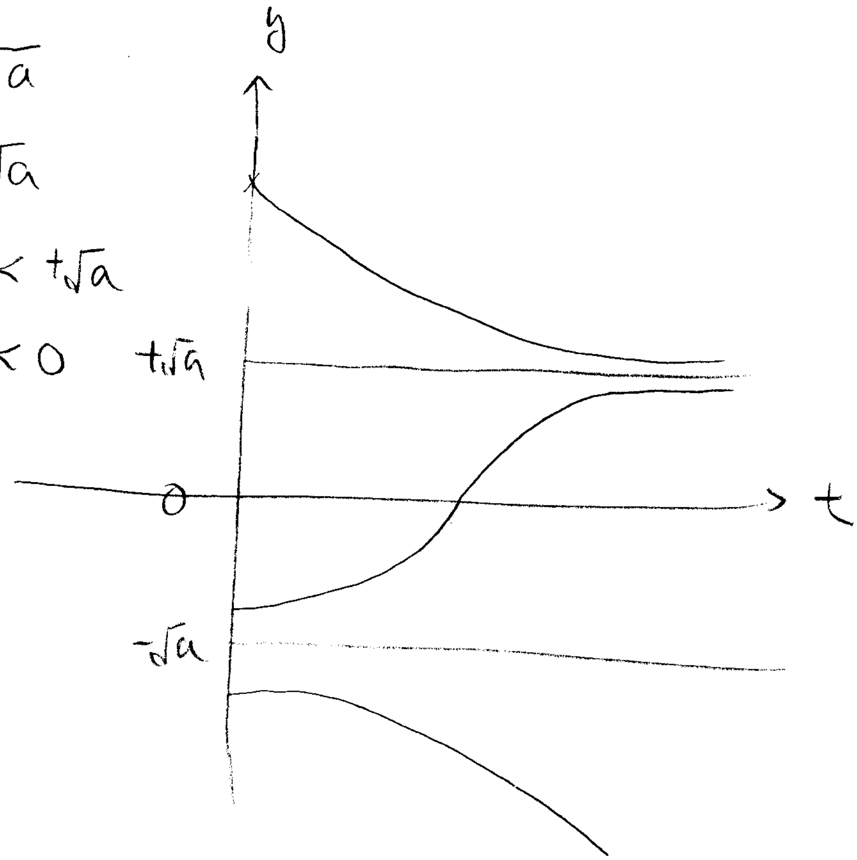
$y'' > 0$ if $-\sqrt{a} < y < 0$

$y > +\sqrt{a}$ concave up

$y < -\sqrt{a}$ concave down

$0 < y < \sqrt{a}$ concave down

$-\sqrt{a} < y < 0$ concave up



§2.6

#1 $(2x+3)+(2y-2)y'=0$

$$M_y = 0 = N_x$$

The equation is exact, Hence

$$\frac{d}{dx}\left(H_1(x)+H_2(y)\right)=0$$

$$H_1(x) = x^2+3x$$

$$H_2(y) = y^2-2y$$

$$\boxed{x^2+3x+y^2-2y = C}$$

#4 $2xy^2+2y+(2x^2y+2x)y'=0$

$$M_y = 4xy+2 = N_x$$

$\psi_x = M = 2xy^2+2y$, $\psi(x,y) = \int (2xy^2+2y)\,dx$

$= x^2y^2+2xy+h(y)$

$$\psi_y = 2x^2y+2x+h'(y) = 2x^2y+2x$$

$h'(y)=0$ $h(y)=0$ ($h(y)$ is any constant c, we chose $c=0$)

$\boxed{x^2y^2+2xy = C}$ is the solution.

#9

$$(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)\,dx + (xe^{xy}\cos 2x - 3)\,dy = 0$$

$$M_y = e^{xy}\cos 2x + xye^{xy}\cos 2x - 2xe^{xy}\sin 2x$$

$$N_x = e^{xy}\cos 2x + xye^{xy}\cos 2x - 2xe^{xy}\sin 2x$$

$$M_y = N_x$$

$$\psi_x = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x = M$$

$$\psi_y = xe^{xy}\cos 2x - 3 = N$$

$$\psi = \int (xe^{xy}\cos 2x - 3)\,dy$$

$$= x \cdot \frac{1}{x} e^{xy}\cos 2x - 3y + h(x)$$

$$= e^{xy}\cos 2x - 3y + h(x)$$

$$\psi_x = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + h'(x) = M$$

$$h'(x) = 2x \qquad h(x) = x^2$$

The solution $\psi = c$ is

$$\boxed{e^{xy}\cos 2x - 3y + x^2 = c}$$

#10.

$$(y/x + 6x)\,dx + (\ln x - 2)\,dy = 0, \qquad x > 0$$

$$M_y = 1/x = N_x$$

$$\psi_x = y/x + 6x \qquad \psi(x,y) = y\ln x + 3x^2 + h(y)$$

$$\psi_y = \ln x - 2 \qquad \frac{\partial\psi}{\partial y} = \ln x + h'(y) = \ln x - 2$$

$$h'(y) = -2 \quad h = -2y$$

$$\psi = C \Rightarrow \boxed{y\ln x + 3x^2 - 2y = C}$$

#16

$$(ye^{2xy} + x)\,dx + bxe^{2xy}\,dy = 0$$

$$\left.\begin{aligned} M_y &= e^{2xy} + 2xye^{2xy} \\ N_x &= be^{2xy} + 2bxye^{2xy} \end{aligned}\right\} M_y = N_x \Rightarrow \boxed{b=1}$$

$$\psi_x = ye^{2xy} + x, \quad \psi(x,y) = y\cdot\frac{1}{2y}e^{2xy} + \frac{1}{2}x^2 + h(y)$$

$$\frac{\partial\psi}{\partial y} = xe^{2xy} + h'(y) = xe^{2xy}$$

$$h'(y) = 0 \qquad h(y) = 0$$

$$\psi = C \Rightarrow \frac{1}{2}e^{2xy} + \frac{1}{2}x^2 = C$$

or $\boxed{e^{2xy} + x^2 = C}$

#21

$$y\,dx + (2x - ye^{y})\,dy = 0 \qquad (1)$$

$$M_y = 1 \quad , \quad N_x = 2 \qquad\qquad M_y \neq N_x$$

$$\mu(x,y) = y \qquad \rightarrow \quad (1)$$

$$y^2\,dx + (2xy - y^2 e^{y})\,dy = 0$$

$$\hat{M} = y^2$$

$$\hat{N} = 2xy - y^2 e^{y}$$

$$\hat{M}_y = 2y \qquad\qquad \psi_x = y^2$$

$$\hat{N}_x = 2y \qquad\qquad \psi(x,y) = xy^2 + h(y)$$

$$\frac{\partial \psi}{\partial y} = 2xy + h'(y) = 2xy - y^2 e^{y}$$

$$h'(y) = -y^2 e^{y}$$

$$h(y) = \int -y^2 e^{y}\,dy = -(2 - 2y + y^2)e^{y}$$

$$\psi = c \Rightarrow \boxed{xy^2 - (2 - 2y + y^2)e^{y} = c}$$

$$\text{or} \quad \boxed{xy^2 - (y^2 - 2y + 2)e^{y} = c}$$

#23

$$(\mu M)_y = (\mu N)_x$$

If $\mu(x,y) = \mu(y)$

$$(\mu N)_x = \mu N_x = \mu_y M + \mu M_y$$

$$\frac{\mu(N_x - M_y)}{M} = \mu_y$$

or
$$\frac{d\mu/dy}{\mu} = \frac{N_x - M_y}{M} = Q(y)$$

Hence
$$\ln|\mu| = \int Q(y)\,dy$$

$$\mu(y) = \exp\left[\int Q(y)\,dy\right]$$

#27 $dx + (x/y - \sin y)dy = 0 \quad - \quad (2)$

$$Q = \frac{N_x - M_y}{M} = \frac{1/y - 0}{1} = \frac{1}{y}$$

$$\mu(y) = \exp\left[\int \frac{1}{y}\,dy\right] = \exp(\ln|y|) = y.$$

$\mu(y) \to$ eq (2)

$$y\,dx + (x - y\sin(y))\,dy = 0$$

$$\psi_x = y, \quad \psi(x,y) = xy + h(y)$$

$$\frac{\partial \psi}{\partial y} = x + h'(y) = x - y\sin(y)$$

$$h'(y) = -y\sin(y)$$

$$h(y) = y\cos(y) - \sin(y)$$

$$\psi(x,y) = C \Rightarrow \boxed{xy + y\cos(y) - \sin(y) = C}$$

#28

$$y\,dx + (2xy - e^{-2y})\,dy = 0 \qquad \text{—— (3)}$$

$$Q = \frac{N_x - M_y}{M} = \frac{2y - 1}{y} = 2 - 1/y$$

$$\mu(y) = \exp\left[\int Q(y)\,dy\right]$$

$$= \exp[2y - \ln|y|]$$

$$= e^{2y}/y$$

$\mu(y) \longrightarrow$ eq (3)

$$e^{2y}dx + \left(2xe^{2y} - \frac{1}{y}\right)dy = 0$$

$$\psi_x = e^{2y} \qquad \psi(x,y) = xe^{2y} + h(y)$$

$$\frac{\partial \psi}{\partial y} = 2xe^{2y} + h'(y) = 2xe^{2y} - \frac{1}{y}$$

$$h'(y) = -\frac{1}{y} \qquad h(y) = -\ln|y|$$

$$\psi = c \rightarrow \boxed{xe^{2y} - \ln|y| = c}$$

#30

$$[4(x^3/y^2) + (3/y)]dx + [3(x/y^2) + 4y]dy = 0$$

$$M_y = -8x^3y^{-3} - 3y^{-2} \qquad \text{— (4)}$$

$$N_x = 3y^{-2}$$

$$Q = \frac{N_x - M_y}{M} = \frac{3y^{-2} + 8x^3y^{-3} + 3y^{-2}}{4(x^3y^{-2}) + 3y^{-1}}$$

$$= \frac{6y^{-2} + 8x^3y^{-3}}{3y^{-1} + 4x^3y^{-2}} = \frac{2y^{-1}(3y^{-1} + 4x^3y^{-2})}{(3y^{-1} + 4x^3y^{-2})}$$

$$Q = \frac{2}{y}$$

$$\frac{d\mu}{\mu} = \frac{2}{y}\, dy$$

$$\mu(y) = \exp\left[\int \frac{2}{y}\, dy\right]$$

$$= \exp[2\ln|y|]$$

$$= e^{\ln(y^2)}$$

$$= y^2$$

$\mu(y) \rightarrow$ eq (4)

$$(4x^3 + 3y)\,dx + (3x + 4y^3)\,dy = 0$$

$$\psi_x = 4x^3 + 3y, \quad \psi(x,y) = x^4 + 3xy + h(y)$$

$$\frac{\partial \psi}{\partial y} = 3x + h'(y) = 3x + 4y^3$$

$$h'(y) = 4y^3 \Rightarrow h(y) = y^4$$

$$\psi = c \Rightarrow \boxed{x^4 + 3xy + y^4 = c}$$