Homework # 5 #1

Sec. 2.8: 2, 5(a), (c), 14

2) $\dfrac{dy}{dt} = 1 - y^3, \; y(-1) = 3 \quad (**)$

Let: $\tau = t + 1; \; \tau = \tau(t); \; \tau(-1) = -1+1 = 0$

$w(\tau(t)) = y(t) - 3 \Rightarrow y(t) = w + 3$

$w(\tau(-1)) = w(0) = y(-1) - 3 = 3 - 3 = 0$

$\dfrac{d}{dt} = \dfrac{d}{d\tau}\dfrac{d\tau}{dt} = \dfrac{d}{d\tau}; \; \dfrac{d\tau}{dt} = \dfrac{d}{dt}(t+1) = 1$

We can rewrite $(**)$ as:

$$\boxed{\dfrac{dw}{d\tau} = 1 - (w+3)^3, \; w(0) = 0}$$

(5) $y' = -\frac{y}{2} + t, \quad y(0) = 0 \quad \underline{(\mathrm{I})}$

For comparison purposes, we solve $\underline{(\mathrm{I})}$ explicitly.

$$\frac{dy}{dt} + \frac{1}{2}y = t, \quad y(0) = 0$$

Int. factor: $e^{\int^t \frac{1}{2}dt'} = e^{t/2}$

$$\underbrace{e^{t/2}\frac{dy}{dt} + \frac{1}{2}e^{t/2}y}_{} = te^{t/2}$$

$$\frac{d}{dt}\left(e^{t/2}y(t)\right) = te^{t/2}$$

$$\int \frac{d}{dt}\left(e^{t/2}y(t)\right)dt = \int te^{t/2}dt$$

$$\Rightarrow e^{t/2}y(t) = 2te^{t/2} - 4e^{t/2} + c$$

$$\Rightarrow y(t) = 2t - 4 + ce^{-t/2}$$

$\underline{y(0)=0}$: $-4 + c = 0 \Rightarrow c = 4$

Homework # 5

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Sec. 2.8

5 $y(t) = 2t - 4 + 4e^{-t/2}$

$y(t) = 2t - 4(1 - e^{-t/2})$

(a) Picard iteration:

$$\frac{dy}{dt} = -\frac{y}{2} + t, \quad y(0) = 0$$

$t_0 = 0, \; y_0 = 0; \; f(t, y) = -\frac{y}{2} + t$

If $y = \phi_n(t)$, then –

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\,ds$$

where $\phi_0 = y_0 = 0$.

$$\phi_{n+1}(t) = \int_0^t \left(s - \frac{\phi_n(s)}{2}\right) ds \quad (n = 0, 1, 2, \ldots)$$

for this problem.

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$\underline{n=0}$: $\phi_1(t) = \int_0^t \left(s - \frac{\phi_0(s)}{2}\right) ds$

$$= \int_0^t s\, ds = \frac{s^2}{2}\Big|_0^t = \frac{t^2}{2}$$

$\underline{n=1}$: $\phi_2(t) = \int_0^t \left(s - \frac{\phi_1(s)}{2}\right) ds$

$$= \int_0^t \left(s - \frac{s^2}{4}\right) ds$$

$$= \left(\frac{s^2}{2} - \frac{s^3}{4\cdot 3}\right)\Big|_0^t = \frac{t^2}{2} - \frac{t^3}{4\cdot 3}$$

$\underline{n=2}$: $\phi_3(t) = \int_0^t \left(s - \frac{\phi_2(s)}{2}\right) ds$

$$= \int_0^t \left(s - \frac{s^2}{4} + \frac{s^3}{4\cdot 3\cdot 2}\right) ds$$

$$= \left(\frac{s^2}{2} - \frac{s^3}{3\cdot 4} + \frac{s^4}{4^2\cdot 3\cdot 2}\right)\Big|_0^t$$

$$= \frac{t^2}{2} - \frac{t^3}{4\cdot 3} + \frac{t^4}{4^2\cdot 3\cdot 2}$$

Homework #5

#3

Sec. 2.8

5 cont'd...

$n=3$: $\phi_4(t) = \int_0^t \left(s - \frac{\phi_3(s)}{2}\right) ds$

$$= \int_0^t \left(s - \frac{s^2}{4} + \frac{s^3}{4\cdot3\cdot2} - \frac{s^4}{4^2\cdot3\cdot2^2}\right) ds$$

$$= \left(\frac{s^2}{2} - \frac{s^3}{4\cdot3} + \frac{s^4}{4^2\cdot3\cdot2} - \frac{s^5}{4^2\cdot5\cdot3\cdot2^2}\right)\Bigg|_0^t$$

$$= \frac{t^2}{2} - \frac{t^3}{4\cdot3} + \frac{t^4}{4^2\cdot3\cdot2} - \frac{t^5}{5\cdot4^2\cdot3\cdot2^2}$$

$\vdots$

$$\phi_n(t) = \frac{t^2}{2} - \frac{t^3}{4\cdot3} + \frac{t^4}{4^2\cdot3\cdot2} - \frac{t^5}{5\cdot4^2\cdot3\cdot2^2} + \cdots + \frac{(-1)^{n+1}t^{n+1}}{2^{n-1}(n+1)!} \quad (n \geq 1)$$

Here: $\phi_1(t), \phi_2(t), \ldots$ represent the sequences of exponential series.

So:

$$\phi_n(t) = \sum_{k=1}^{n} \frac{(-1)^{k+1}t^{k+1}}{2^{k-1}(k+1)!}$$

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(c) Determine $\lim_{n\to+\infty} \phi_n(t)$.

$$\lim_{n\to+\infty} \phi_n(t) = \sum_{q=1}^{+\infty} \frac{(-1)^{q+1} t^{q+1}}{2^{q-1}(q+1)!}$$

Let: $k = q+1 \Rightarrow q = k-1$

$1 \le q < +\infty \Rightarrow 2 \le k < +\infty$

$$\sum_{q=1}^{+\infty} \frac{(-1)^{q+1} t^{q+1}}{2^{q-1}(q+1)!} = \sum_{k=2}^{+\infty} \frac{(-1)^{k} t^{k}}{2^{k-2}\, k!}$$

$$= \frac{1}{2^{-2}} \sum_{k=2}^{+\infty} \frac{(-t/2)^k}{k!} = 2^2 \sum_{k=2}^{+\infty} \frac{(-t/2)^k}{k!}$$

But: $$e^{-t/2} = \sum_{k=0}^{+\infty} \frac{(-t/2)^k}{k!}$$

$$= 1 - \frac{t}{2} + \sum_{k=2}^{+\infty} \frac{(-t/2)^k}{k!}$$

∴ Conclusion:

$$\boxed{\begin{aligned}\lim_{n\to+\infty} \phi_n(t) &= 4 \sum_{k=2}^{+\infty} \frac{(-t/2)^k}{k!} \\ &= 4\left(e^{-t/2} - 1 + \frac{t}{2}\right) \\ &= 2t - 4 + 4e^{-t/2}\end{aligned}}$$

(agrees w/ exact solution)

Homework #5 #7

Sec. 5.8

(14) Let: $\phi_n(x) = 2nxe^{-nx^2}, \ 0 \le x \le 1$

(a) Show that $\lim_{n\to+\infty} \phi_n(x) = 0, \ 0 \le x \le 1.$

$x=0$: $\phi_n(0) = 0; \ \lim_{n\to+\infty} \phi_n(0) = 0$ ☐

$x=1$: $\phi_n(1) = 2ne^{-n};$

$$\lim_{n\to+\infty} \phi_n(1) = \lim_{n\to+\infty} \frac{2n}{e^n} = 0 \ ☐$$

$0<x<1$: $\phi_n(x) = 2nxe^{-nx^2}$

$$\lim_{n\to+\infty} \phi_n(x) = \lim_{n\to+\infty} \frac{2nx}{e^{nx^2}} = 0 \quad (0<x<1)$$

Therefore: $\lim_{n\to+\infty} \phi_n(x) = 0$ for $0 \le x \le 1$.

$$\int_0^1 \lim_{n\to+\infty} \phi_n(x)\,dx = \int_0^1 0\,dx = \underline{\underline{0}} \ ☐$$

(b) $\int_0^1 \phi_n(x)\,dx = \underbrace{\int_0^1 2nxe^{-nx^2}\,dx}$

$u = -nx^2,\ du = -2nx\,dx$

$$\int_0^1 2nxe^{-nx^2}\,dx = -\int_0^{-n} e^u\,du = -e^u\Big|_0^{-n}$$

$$\Rightarrow \int_0^1 \phi_n(x)\,dx = \int_0^1 2nxe^{-nx^2}\,dx = 1 - e^{-n}$$

So –

$$\lim_{n\to+\infty}\left(\int_0^1 \phi_n(x)\,dx\right) = \lim_{n\to+\infty}(1-e^{-n}) = \underline{\underline{1}}.$$

Thus we have:

$$\lim_{n\to+\infty}\int_0^1 \phi_n(x)\,dx \neq \int_0^1\left(\lim_{n\to+\infty}\phi_n(x)\right)dx.$$

Homework 8 #5 #9

Sec. 3.1: 4, 7, 11, 13, 21

4) $2y'' - 3y' + y = 0$

Let: $y = e^{rt}$

$(2r^2 - 3r + 1)e^{rt} = 0 \Rightarrow 2r^2 - 3r + 1 = 0$

$\Rightarrow r = \dfrac{3 \pm \sqrt{9 - 4(2)(1)}}{4} = \dfrac{3 \pm \sqrt{1}}{4} = \dfrac{3 \pm 1}{4}$

$\Rightarrow r_1 = \dfrac{3+1}{4} = 1, \; r_2 = \dfrac{3-1}{4} = \dfrac{1}{2}$

So: $y_1(t) = e^{r_1 t} = e^{t}$

$y_2(t) = e^{r_2 t} = e^{t/2}$

$\boxed{\text{General soln: } y(t) = c_1 e^{t} + c_2 e^{t/2}}$

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(9) $y'' - 9y' + 9y = 0$

Let: $y = e^{rt}$

$(r^2 - 9r + 9)e^{rt} = 0 \Rightarrow r^2 - 9r + 9 = 0$

$r = \frac{9 \pm \sqrt{81-36}}{2} = \frac{9 \pm \sqrt{45}}{2} = \frac{9 \pm 3\sqrt{5}}{2}$

$\Rightarrow r_1 = \frac{9+3\sqrt{5}}{2}, \quad r_2 = \frac{9-3\sqrt{5}}{2}$

$y_1(t) = e^{r_1 t} = e^{\left(\frac{9+3\sqrt{5}}{2}\right)t}, \quad y_2(t) = e^{r_2 t} = e^{\left(\frac{9-3\sqrt{5}}{2}\right)t}$

General sol^n:

$$\boxed{y(t) = c_1 e^{\left(\frac{9+3\sqrt{5}}{2}\right)t} + c_2 e^{\left(\frac{9-3\sqrt{5}}{2}\right)t}}$$

Homework #5

#6 .11

(11) $6y'' - 5y' + y = 0$, $y(0) = 4$, $y'(0) = 0$

Let: $y = e^{rt}$; $(6r^2 - 5r + 1)e^{rt} = 0$

$\Rightarrow 6r^2 - 5r + 1 = 0$

$$r = \frac{5 \pm \sqrt{25-24}}{12} = \frac{5 \pm 1}{12}$$

$\Rightarrow r_1 = \frac{5+1}{12} = \frac{6}{12} = \frac{1}{2}$, $r_2 = \frac{5-1}{12} = \frac{4}{12} = \frac{1}{3}$

$y_1(t) = e^{r_1 t} = e^{t/2}$, $y_2(t) = e^{r_2 t} = e^{t/3}$

So: $y(t) = c_1 e^{t/2} + c_2 e^{t/3}$

$y'(t) = \frac{c_1}{2} e^{t/2} + \frac{c_2}{3} e^{t/3}$

$y(0) = 4$: $c_1 + c_2 = 4$

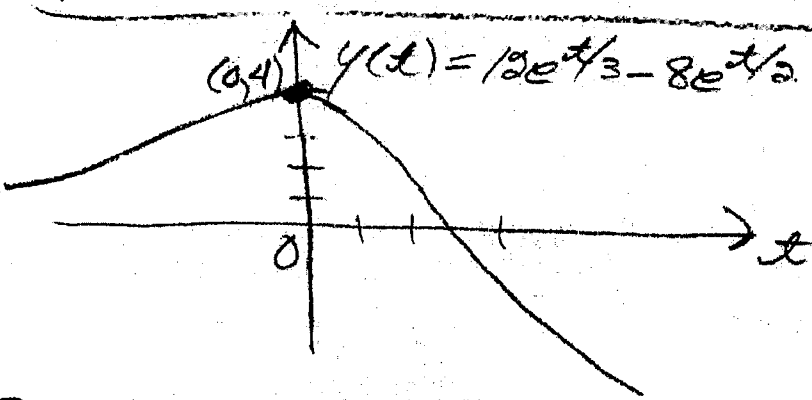
$y'(0) = 0$: $\frac{c_1}{2} + \frac{c_2}{3} = 0 \Rightarrow \frac{c_2}{3} = -\frac{c_1}{2} \Rightarrow c_2 = -\frac{3}{2} c_1$

Now:

$c_1 + c_2 = c_1 - \frac{3}{2} c_1 = -\frac{c_1}{2} = 4 \Rightarrow c_1 = -8$

$\Rightarrow c_2 = -\frac{3}{2} c_1 = -\frac{3}{2}(-8) = \underline{12}$

$$\therefore \boxed{y(t) = -8e^{t/2} + 12e^{t/3}}$$



(13) $y'' + 5y' + 3y = 0, \quad y(0) = 1, \quad y'(0) = 0$

Let: $y = e^{rt}$; $(r^2 + 5r + 3)e^{rt} = 0$

$\Rightarrow r^2 + 5r + 3 = 0$

$$r = \frac{-5 \pm \sqrt{25 - 12}}{2} = \frac{-5 \pm \sqrt{13}}{2}$$

$$\Rightarrow r_1 = \frac{-5 + \sqrt{13}}{2}, \quad r_2 = \frac{-5 - \sqrt{13}}{2}$$

So:

$$y_1(t) = e^{r_1 t} = e^{\left(\frac{-5+\sqrt{13}}{2}\right)t}$$

$$y_2(t) = e^{r_2 t} = e^{\left(\frac{-5-\sqrt{13}}{2}\right)t}$$

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(13) cont'd...

$$y(t) = c_1 e^{\left(\frac{-5+\sqrt{13}}{2}\right)t} + c_2 e^{\left(\frac{-5-\sqrt{13}}{2}\right)t}$$

$$y'(t) = c_1\left(\frac{-5+\sqrt{13}}{2}\right) e^{\left(\frac{-5+\sqrt{13}}{2}\right)t} + c_2\left(\frac{-5-\sqrt{13}}{2}\right) e^{\left(\frac{-5-\sqrt{13}}{2}\right)}$$

$\underline{y(0)=1}$: $c_1 + c_2 = 1$

$\underline{y'(0)=0}$: $\left(\frac{-5+\sqrt{13}}{2}\right)c_1 + \left(\frac{-5-\sqrt{13}}{2}\right)c_2 = 0$

$$\Rightarrow c_2 = \left(\frac{-\cancel{2}}{5+\sqrt{13}}\right)\left(\frac{5-\sqrt{13}}{\cancel{2}}\right)c_1$$

$$\Rightarrow c_2 = \left(\frac{-5+\sqrt{13}}{5+\sqrt{13}}\right)c_1$$

$\underline{\text{So}}$: $c_1 + c_2 = c_1 + \left(\frac{-5+\sqrt{13}}{5+\sqrt{13}}\right)c_1 = \frac{2\sqrt{13}}{5+\sqrt{13}}c_1 = 1$

$$\Rightarrow c_1 = \frac{5+\sqrt{13}}{2\sqrt{13}} \quad \Rightarrow c_2 = \left(\frac{-5+\sqrt{13}}{\cancel{5+\sqrt{13}}}\right)\left(\frac{\cancel{5+\sqrt{13}}}{2\sqrt{13}}\right)$$

$$\Rightarrow c_2 = \frac{-5+\sqrt{13}}{2\sqrt{13}}$$

$$\therefore \boxed{y(t) = \left(\frac{5+\sqrt{13}}{2\sqrt{13}}\right)e^{\left(\frac{-5+\sqrt{13}}{2}\right)t} + \left(\frac{-5+\sqrt{13}}{2\sqrt{13}}\right)e^{\left(\frac{-5-\sqrt{13}}{2}\right)t}}$$