HW # 6

#1

Sec. 3.2: 3, 6, 8, 10, 18

(3) $f(t) = e^{-2t}$, $g(t) = te^{-2t}$

$f'(t) = -2e^{-2t}$, $g'(t) = e^{-2t} - 2te^{-2t}$

$$= e^{-2t}(1-2t)$$

Now:

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t}(1-2t) \end{vmatrix}$$

$$\Rightarrow W(f,g) = e^{-4t}(1-2t) + 2te^{-4t}$$

$$= e^{-4t} - 2te^{-4t} + 2te^{-4t}$$

$$\therefore \boxed{W(f,g) = e^{-4t}}$$

(#2)

Homework #6

Sec. 3.2

(6) $f(\theta) = \cos^2\theta$, $g(\theta) = 1+\cos(2\theta)$

$f'(\theta) = -2\cos\theta\sin\theta$, $g'(\theta) = -2\sin(2\theta)$

Now:

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} \cos^2\theta & 1+\cos(2\theta) \\ -2\cos\theta\sin\theta & -2\sin(2\theta) \end{vmatrix}$$

$$= -2\cos^2\theta\sin(2\theta) + 2\cos\theta\sin\theta \times (1+\cos(2\theta))$$

Use the trig identities:

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\sin(2\theta) = 2\cos(\theta)\sin(\theta)$$

to determine $W(f,g)$.

$$W(f,g) = -2\cos^2\theta\sin(2\theta) + (2\cos\theta\sin\theta) \times (1+\cos(2\theta))$$

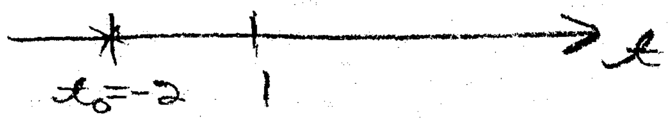
$$= -(1+\cos(2\theta))(\sin 2\theta) + \sin 2\theta(1+\cos(2\theta))$$

$\therefore$ $\boxed{W(f,g) = 0}$

(8) $(t-1)y'' - 3ty' + 4y = \sin(t)$,
$y(-2) = 2$, $y'(-2) = 1$

<u>So</u>: $y'' - \dfrac{3t}{t-1}y' + \dfrac{4}{t-1}y = \dfrac{\sin t}{t-1}$

<u>$t \neq 1$</u> since $t-1$ is in the denominator. Here, <u>$t_0 = -2$</u>.

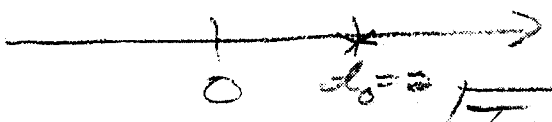


$$\boxed{\text{Interval of Sol'n: } -\infty < t < 1}$$

(10) $y'' + (\cos t)y' + 3(\ln|t|)y = 0$
$y(2) = 3$, $y'(2) = 1$

$\cos t$ – defined for $-\infty < t < +\infty$
$\ln|t|$ – not defined at <u>$t=0$</u>

Here, <u>$t_0 = 2$</u>.



$$\boxed{\text{Interval: } 0 < t < +\infty}$$

Homework #6 (#4)

Sec. 3.2

(18) If $W(f,g) = t^2e^t$ and if $f(t) = t$, determine $g(t)$.

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - gf'$$

So:

$$W(f,g) = fg' - gf' = tg' - g(1) = t^2e^t$$

$$\Rightarrow tg' - g = t^2e^t$$

— or —

$$\boxed{g' - \frac{1}{t}g = te^t} \quad (*)$$

We solve (*) for $g(t)$ via integrating factor.

$$\mu(t) = e^{-\int^t \frac{dt'}{t'}} = e^{-\ln t} = t^{-1}$$

Now:

$$\frac{1}{t}g' - \frac{1}{t^2}g = e^t$$

#5

$$\int \frac{d}{dt}\left(\frac{g(t)}{t}\right)dt = \int e^{t}\,dt$$

$$\Rightarrow \frac{g(t)}{t} = e^{t} + c$$

$$\Rightarrow \boxed{g(t) = te^{t} + ct}$$

Homework # 6

#6

Sec. 3.3: 5, 6, 7, 16, 17

5 $f(t) = 3t-5,\ g(t) = 9t-15$

$f'(t) = 3,\ g'(t) = 9$

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} 3t-5 & 9t-15 \\ 3 & 9 \end{vmatrix}$$

$$\Rightarrow W(f,g) = 9(3t-5) - 3(9t-15)$$

$$= 27t - 45 - 27t + 45 = \underline{\underline{0}}$$

$\Rightarrow$ $f(t) = 3t-5$ and $g(t) = 9t-15$ are linearly dependent

$\left(g(t) = 9t-15 = 3f(t) \right)$

#

(b) $f(t) = t, \quad g(t) = t^{-1}$

$$f'(t) = 1, \quad g'(t) = -t^{-2}$$

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} t & t^{-1} \\ 1 & -t^{-2} \end{vmatrix}$$

$$\Rightarrow W(f,g) = -t^{-2}(t) - t^{-1} = -t^{-1} - t^{-1}$$
$$= -2t^{-1}$$

If $t \neq 0$, then $W(f,g) = -2t^{-1} \neq 0$, so $f(t)$ and $g(t)$ are linearly independent.

Homework #6

#8

Sec. 3.3:

(9) $W(t) = t\sin^2 t$

The functions are linearly independent since $W(t) \neq 0$ for all t.

(16) $(\cos t)y'' + (\sin t)y' - ty = 0$

$\Rightarrow y'' + (\tan t)y' - \left(\frac{t}{\cos t}\right)y = 0$

Identify: $p(t) = \tan t$

$$W(t) = Ae^{-\int^t p(t')dt'} = Ae^{-\int^t \left(\frac{\sin t'}{\cos t'}\right)dt'}$$

$$= Ae^{+\ln|\cos t|} = A(\cos t)$$

$\therefore$ $\boxed{W(t) = A\cos t}$

#:

[redacted]

(17) $x^2 y'' + x y' + (x^2 - \nu^2) y = 0, \quad x > 0$

Sol:

$$y'' + \frac{1}{x} y' + \frac{(x^2 - \nu^2)}{x^2} y = 0$$

Identify: $p(x) = \frac{1}{x}$

$$W(x) = A e^{-\int^x p(x')\,dx'} = A e^{-\int^x \frac{dx'}{x'}}$$

$$= A e^{-\ln|x|} = A x^{-1}$$

$$\therefore \boxed{W(x) = A x^{-1} = A/x}$$