Homework #7 (#1)

Sec. 3.4: 1, 5, 11, 17, 22, 27, 29

(1) $e^{1+2i} = e^{1} \cdot e^{i2}$

$= e(\cos(2) + i\sin(2))$

$\therefore \boxed{e^{1+2i} = e\cos(2) + ie\sin(2)}$

Euler's Identity: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

(5) $2^{1-i} = e^{\ln[2^{(1-i)}]} = e^{(1-i)\ln(2)}$

$= e^{\ln(2) - i\ln(2)}$

$= e^{\ln(2)} e^{-i\ln(2)}$

$= e^{\ln(2)}[\cos(-\ln(2)) + i\sin(-\ln(2))]$

$\Rightarrow \boxed{2^{1-i} = 2\cos[\ln(2)] - i2\sin[\ln(2)]}$

#2

(11) $y'' + 6y' + 13y = 0$

**Let:** $y = e^{rt}$

$$(r^2 + 6r + 13)e^{rt} = 0 \Rightarrow r^2 + 6r + 13 = 0$$

$$\Rightarrow r = \frac{-6 \pm \sqrt{36 - 52}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i$$

**Identify:** $r = \lambda \pm i\mu = -3 \pm 2i$

$$\Rightarrow \underline{\lambda = -3}, \ \underline{\mu = 2}$$

**General sol'n:**

$$\boxed{y(t) = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t)}$$

## Homework #7

#3

(17) $y'' + 4y = 0,\ y(0) = 0,\ y'(0) = 1$

Let: $y = e^{rt}$

$(r^2+4)e^{rt} = 0 \Rightarrow r^2 + 4 = 0 \Rightarrow r = \pm 2i$

Identify: $r = \lambda \pm i\mu = \pm 2i$

$\Rightarrow \lambda = 0,\ \mu = 2$

Solution:

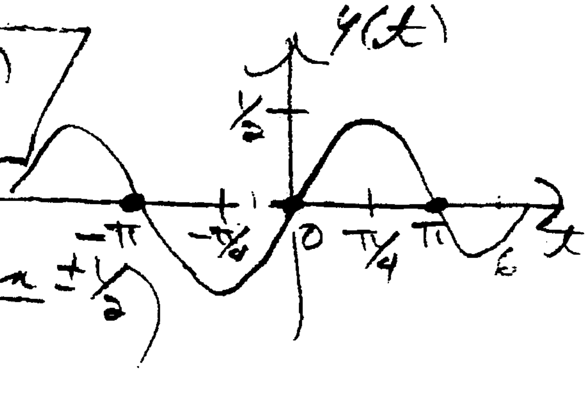
$y(t) = c_1 \cos(2t) + c_2 \sin(2t)$

$y'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t)$

$y(0) = 0$: $c_1 = 0$

$y'(0) = 1$: $2c_2 = 1 \Rightarrow c_2 = 1/2$

$\therefore \boxed{y(t) = \frac{1}{2}\sin(2t)}$




($y(t)$ oscillates between $\pm\frac{1}{2}$ as $t \to +\infty$.)

[redacted] #4

QVP

22 $y''+2y'+2y=0, \ y(\pi/4)=2, \ y'(\pi/4)=-2$

Let: $y=e^{rt}$

$(r^2+2r+2)e^{rt}=0 \Rightarrow r^2+2r+2=0$

$$r=\frac{-2\pm\sqrt{4-8}}{2}=\frac{-2\pm 2i}{2}=-1\pm i$$

Identify: $r=\lambda\pm i\mu=-1\pm i$

$\Rightarrow \underline{\lambda=-1}, \underline{\mu=1}$

Solution:

$$y(t)=c_1e^{-t}\cos(t)+c_2e^{-t}\sin(t)$$

$$y'(t)=-c_1e^{-t}\cos t-c_1e^{-t}\sin(t)$$
$$-c_2e^{-t}\sin t+c_2e^{-t}\cos(t)$$

Now:

$y(\pi/4)=2$: $c_1e^{-\pi/4}\left(\frac{\sqrt{2}}{2}\right)+c_2e^{-\pi/4}\left(\frac{\sqrt{2}}{2}\right)=2$

$\Rightarrow e^{-\pi/4}\sqrt{2}(c_1+c_2)=4$

$$\Rightarrow c_1+c_2=\frac{4}{\sqrt{2}e^{-\pi/4}}=\frac{4e^{\pi/4}}{\sqrt{2}}$$

# Homework #7

## Sec. 3.4: (cont'd...)

22 $\underline{y'(\pi/4) = -2}$: $-c_1 e^{-\pi/4}\left(\frac{\sqrt{2}}{2}\right) - c_1 e^{-\pi/4}\left(\frac{\sqrt{2}}{2}\right)$

$$- c_2 \cancel{e^{-\pi/4}\left(\frac{\sqrt{2}}{2}\right)} + c_2 \cancel{e^{-\pi/4}\left(\frac{\sqrt{2}}{2}\right)} = -2$$

$\Rightarrow -c_1 e^{-\pi/4}\sqrt{2} = -2$

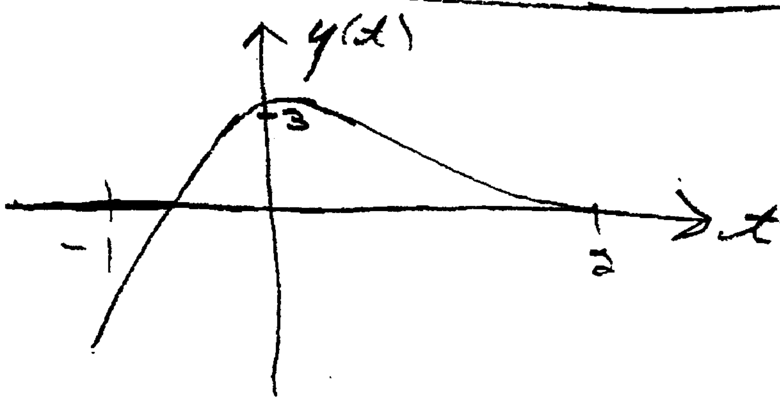
$\Rightarrow$ $c_1 = \frac{2}{\sqrt{2}\,e^{-\pi/4}} = \frac{2e^{\pi/4}}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}} e^{\pi/4} = \sqrt{2}\,e^{\pi/4}$

$\underline{\text{Recall}}$: $c_1 + c_2 = \frac{2e^{\pi/4}}{\sqrt{2}} + c_2 = \frac{4e^{\pi/4}}{\sqrt{2}} \Rightarrow c_2 = \frac{2}{\sqrt{2}} e^{\pi/4}$

$\therefore$ $$\boxed{y(t) = \frac{2e^{\pi/4}}{\sqrt{2}} e^{-t}\cos(t) + \frac{2e^{\pi/4}}{\sqrt{2}} e^{-t}\sin(t)}$$

– or –

$$\boxed{y(t) = \sqrt{2}\,e^{-(t-\pi/4)}\cos(t) + \sqrt{2}\,e^{-(t-\pi/4)}\sin(t)}$$



Homework #7

#6

(27) $f(t) = e^{\lambda t}\cos(\mu t)$
$g(t) = e^{\lambda t}\sin(\mu t)$

Determine: $W(f,g)$.

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - gf'$$

Here:

$$f'(t) = \lambda e^{\lambda t}\cos(\mu t) - \mu e^{\lambda t}\sin(\mu t)$$

$$g'(t) = \lambda e^{\lambda t}\sin(\mu t) + \mu e^{\lambda t}\cos(\mu t)$$

So:

$$W(f,g) = fg' - gf'$$
$$= \cancel{\lambda e^{2\lambda t}\sin(\mu t)\cos(\mu t)}$$
$$+ \mu e^{2\lambda t}\cos^2(\mu t) - \cancel{\lambda e^{2\lambda t}\sin(\mu t)\cos(\mu t)}$$
$$+ \mu e^{2\lambda t}\sin^2(\mu t)$$

$$\therefore W(f,g) = \mu e^{2\lambda t}[\underbrace{\cos^2(\mu t) + \sin^2(\mu t)}_{=1}]$$

$$\boxed{W(f,g) = \mu e^{2\lambda t}}$$

Homework #7

#7

(29) Show that -

$$\cos t = \frac{e^{it} + e^{-it}}{2}$$

$$\sin t = \frac{e^{it} - e^{-it}}{2i}.$$

Pf: Use $e^{\pm it} = \cos(t) \pm i\sin(t)$

(i) $$\frac{e^{it} + e^{-it}}{2} = \frac{1}{2}\left[\cos t + i\sin t + \cos t - i\sin t\right]$$
$$= \frac{1}{2}(2\cos t) = \cos t \boxtimes$$

(ii) $$\frac{e^{it} - e^{-it}}{2i} = \frac{1}{2i}\left[\cos t + i\sin t - \cos t + i\sin t\right]$$
$$= \frac{1}{2i}(2i\sin t) = \sin t \boxtimes$$

Homework # 7 #8

Sec. 3.5: 1, 6, 11, 20(a)-(c)

1 $y'' - 2y' + y = 0$

Let: $y = e^{rt}$

$(r^2 - 2r + 1)e^{rt} = 0 \Rightarrow r^2 - 2r + 1 = 0$

$r^2 - 2r + 1 = (r-1)^2 = 0 \Rightarrow r - 1 = 0$

So: $\Rightarrow r = 1$

$y_1(t) = e^{rt} = e^t$

$y_2(t) = te^t$

General sol'n:

$$\boxed{y(t) = c_1 e^t + c_2 t e^t}$$

Homework # 7

#9

(6) $y''-6y'+9y=0$

Let: $y=e^{rt}$

$(r^2-6r+9)e^{rt}=0 \Rightarrow r^2-6r+9=0$

$r^2-6r+9=(r-3)^2=0 \Rightarrow r-3=0$

$\Rightarrow \underline{r=3}$

So:

$y_1(t)=e^{rt}=e^{3t}$

$y_2(t)=te^{3t}$

General Sol'n:

$$\boxed{y(t)=c_1e^{3t}+c_2te^{3t}}$$

Homework # 7 

11 $9y'' - 12y' + 4y = 0,\ y(0) = 2,\ y'(0) = -1$

Let: $y = e^{rt}$

$(9r^2 - 12r + 4)e^{rt} = 0$

$\Rightarrow 9r^2 - 12r + 4 = 0 \Rightarrow r = \dfrac{12 \pm \sqrt{144 - 144}}{18}$

$\Rightarrow r = \dfrac{12}{18} = \dfrac{2}{3}$

So: $y_1(t) = e^{rt} = e^{2t/3}$

$y_2(t) = t e^{2t/3}$

Solution: $y(t) = c_1 e^{2t/3} + c_2 t e^{2t/3}$

$y'(t) = \frac{2}{3} c_1 e^{2t/3} + c_2 e^{2t/3} + \frac{2}{3} c_2 t e^{2t/3}$

$y(0) = 2$: $c_1 = 2$

$y'(0) = -1$: $\frac{2}{3} c_1 + c_2 = -1 \Rightarrow c_2 = -1 - \frac{2}{3} c_1 = -1 - \frac{4}{3} = -\frac{7}{3}$

$\therefore \boxed{y(t) = 2e^{2t/3} - \frac{7}{3} t e^{2t/3}}$ $\left( \begin{array}{l} y \to -\infty \\ \text{as } t \to +\infty \end{array} \right)$

# Homework # 7

#11

(20) (a) $y'' + 2ay' + a^2 y = 0$ (I)

Let: $y = e^{rt}$

$(r^2 + 2ar + a^2) e^{rt} = 0$

$\Rightarrow r^2 + 2ar + a^2 = (r+a)^2 = 0$

$\Rightarrow r + a = 0 \Rightarrow r = -a$

$$\therefore \boxed{y_1(t) = e^{rt} = e^{-at}}$$

(b) Let $y_1(t)$ and $y_2(t)$ be two solutions to (I). Then:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

From Abel's Thm (Sec. 3.3):

For: $y'' + p(t) y' + q(t) y = 0$

$$W(y_1, y_2) = A e^{-\int^t p(t')dt'} \quad (A\text{-constant})$$

## Homework #1



20 (b) cont'd...

For this problem, $p(t) = 2a$. So:

$$W(y_1, y_2) = A e^{-\int^t (2a)\,dt'} = A e^{-2at}$$

$$\therefore \boxed{y_1 y_2' - y_2 y_1' = A e^{-2at}}$$

(c) Determine $y_2(t)$ from (a) and (b).

$y_1(t) = e^{-at}$, $y_1'(t) = -a e^{-at}$

$$y_1 y_2' - y_2 y_1' = A e^{-2at}$$

$$(e^{-at}) y_2' - (-a e^{-at}) y_2 = A e^{-2at}$$

— or —

$$\boxed{y_2' + a y_2 = A e^{-at}}$$ — We must solve this for $y_2(t)$!

Homework #7

#13

(20) (c) cont'd...

$$y_2' + a y_2 = A e^{-at}$$

Int. factor: $e^{\int^t a\,dt'} = e^{at}$

$$\underbrace{e^{at} y_2' + a e^{at} y_2}_{} = A$$

$$\int \frac{d}{dt}\left(e^{at} y_2(t)\right) dt = \int A\,dt = At + B$$

$$\Rightarrow e^{at} y_2(t) = (At + B)$$

— or —

$$y_2(t) = (At + B) e^{-at}$$

$$\boxed{\begin{array}{l}\text{Linearly independent solutions:}\\ y_1(t) = e^{-at},\quad y_2(t) = t e^{-at}\end{array}}$$