Homework #8

#1

Sec. 3.6: 2, 3, 7, 9, 17

(2) $y'' + 2y' + 5y = 3\sin(2t)$

Homogeneous piece:

$y'' + 2y' + 5y = 0$

Let: $y = e^{rt}$: $r^2 + 2r + 5 = 0$

$r = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

So:

$y_H(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$

Particular piece:

$g(t) = 3\sin(2t)$

Let: $y_P(t) = A\sin(2t) + B\cos(2t)$

$y_P'(t) = 2A\cos(2t) - 2B\sin(2t)$

$y_P''(t) = -4A\sin(2t) - 4B\cos(2t)$

#2

[redacted]

Now: $y_p'' + 2y_p' + 5y_p = 3\sin(2t)$

$$-4A\sin(2t) - 4B\cos(2t)$$
$$+4A\cos(2t) - 4B\sin(2t)$$
$$+5A\sin(2t) + 5B\cos(2t) = 3\sin(2t)$$

$$\Rightarrow (A-4B)\sin(2t) + (B+4A)\cos(2t)$$
$$= 3\sin(2t)$$

$\Rightarrow A - 4B = 3$ and $B + 4A = 0$

$\Rightarrow \underline{B = -4A}$

$$A - 4B = A - 4(-4A) = 17A = 3 \Rightarrow A = \frac{3}{17}$$

$$B = -4A = -\frac{12}{17}$$

$$y_p(t) = -\frac{3}{15}\sin(2t) + \frac{12}{15}\cos(2t)$$

$$\therefore \boxed{y(t) = y_H(t) + y_p(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t) + \frac{3}{17}\sin(2t) - \frac{12}{17}\cos(2t)}$$

Homework #8

#3

③ $y'' - 2y' - 3y = -3te^{-t}$

Homogeneous piece:

$y'' - 2y' - 3y = 0$

Let: $y = e^{rt}$; $r^2 - 2r - 3 = (r-3)(r+1) = 0$

$\Rightarrow r = -1, 3$

$y_H(t) = c_1 e^{-t} + c_2 e^{3t}$

Particular piece:

$g(t) = -3te^{-t}$

Let:

$y_p(t) = (At^2 + Bt)e^{-t}$

Q: Why this form for $y_p(t)$?

A: If we take $y_p(t) = Ate^{-t}$, then –

$y_p' = Ae^{-t} - Ate^{-t}$

$y_p'' = -2Ae^{-t} + Ate^{-t}$

Homework #8 $\quad$ #4

(3) cont'd...

$$y_p'' - 2y_p' - 3y_p = -3te^{-t}$$

$$\Rightarrow -2Ae^{-t} + Ate^{-t} - 2Ae^{-t} + 2Ate^{-t} - 3Ate^{-t} = -3te^{-t}$$

$$\Rightarrow -4Ae^{-t} = -3te^{-t}$$ – not possible for A constant!

So our assumption $y_p(t) = Ate^{-t}$ fails here! //

Take: $y_p(t) = (At^2 + Bt)e^{-t}$

$$y_p'(t) = (2At + B)e^{-t} - (At^2 + Bt)e^{-t}$$

$$y_p''(t) = -2(2At + B)e^{-t} + (At^2 + Bt)e^{-t} + 2Ae^{-t}$$

3 cont'd...

So: $y_p'' - 2y_p' - 3y_p = -3te^{-t}$

$$2Ae^{-t} - 4Ate^{-t} - 2Be^{-t} + \cancel{At^2e^{-t}}$$
$$+ Bte^{-t} - 4Ate^{-t} - 2Be^{-t} + \cancel{2At^2e^{-t}}$$
$$+ 2Bte^{-t} - \cancel{3At^2e^{-t}} - 3Bte^{-t} = -3te^{-t}$$

$\Rightarrow (2A - 4B)e^{-t} - 8Ate^{-t} = -3te^{-t}$

$\Rightarrow 2A - 4B = 0$ and $-8A = -3 \Rightarrow A = \frac{3}{8}$

$2A - 4B = 0 \Rightarrow 4B = 2A \Rightarrow B = \frac{A}{2} = \frac{3}{16}$

So:

$$y_p(t) = \frac{3}{8}t^2e^{-t} + \frac{3}{16}te^{-t}$$

$$\therefore \boxed{\begin{aligned} y(t) &= y_H(t) + y_p(t) \\ &\text{-or-} \\ y(t) &= c_1e^{-t} + c_2e^{3t} + \frac{3}{8}t^2e^{-t} + \frac{3}{16}te^{-t} \end{aligned}}$$

#6

(9) $2y'' + 3y' + y = t^2 + 3\sin(t)$

$$y'' + \frac{3}{2}y' + \frac{1}{2}y = \frac{t^2}{2} + \frac{3}{2}\sin(t)$$

(coefficient multiplying y'' term must be <u>1</u>)

<u>Homogeneous piece:</u>

$$y'' + \frac{3}{2}y' + \frac{1}{2}y = 0$$

<u>Let:</u> $y = e^{rt}$; $r^2 + \frac{3}{2}r + \frac{1}{2} = 0$

$$\Rightarrow r = \frac{-3/2 \pm \sqrt{9/4 - 4/2}}{2} = \frac{-\frac{3}{2} \pm \sqrt{\frac{1}{4}}}{2} = -\frac{3}{4} \pm \frac{1}{4}$$

$$r = -1/2 \ \underline{\text{or}} \ -1$$

$$y_H(t) = c_1 e^{-t/2} + c_2 e^{-t}$$

<u>Particular piece:</u>

$$g(t) = \frac{t^2}{2} + \frac{3}{2}\sin(t)$$

<u>Assume:</u>

$$y_p(t) = At^2 + Bt + C + D\sin(t) + E\cos(t)$$

Homework #8 (#7)

(7) cont'd...

We have to solve for A, B, C, D and E.

Now: $y_p(t) = At^2 + Bt + C + D\sin(t) + E\cos(t)$

$y_p'(t) = 2At + B + D\cos(t) - E\sin(t)$

$y_p''(t) = 2A - D\sin(t) - E\cos(t)$

So: $y_p'' + \frac{3}{2} y_p' + \frac{1}{2} y_p = \frac{t^2}{2} + \frac{3}{2}\sin(t)$

$\Rightarrow 2A - D\sin(t) - E\cos(t) + 3At + \frac{3}{2}B + \frac{3}{2}D\cos(t) - \frac{3}{2}E\sin(t) + \frac{1}{2}At^2 + \frac{1}{2}Bt + \frac{C}{2} + \frac{D}{2}\sin(t) + \frac{E}{2}\cos(t) = \frac{t^2}{2} + \frac{3}{2}\sin(t)$

[redacted]

$$\Rightarrow \left(2A + \frac{3}{2}B + \frac{C}{2}\right) + \left(3A + \frac{B}{2}\right)t$$

$$+ \frac{A}{2}t^2 - \left(\frac{D}{2} + \frac{3E}{2}\right)\sin(t)$$

$$+ \left(-\frac{E}{2} + \frac{3D}{2}\right)\cos(t) = \frac{t^2}{2} + \frac{3}{2}\sin(t)$$

<u>Matching coefficients</u>:

<u>t^2-terms</u>: $\frac{A}{2} = \frac{1}{2} \Rightarrow A = 1$

<u>t-terms</u>: $3A + \frac{B}{2} = 0 \Rightarrow B = -6A = -6$

<u>Constants</u>: $2A + \frac{3}{2}B + \frac{C}{2} = 0$

$$\Rightarrow C = -2\left(2A + \frac{3}{2}B\right)$$

$$\Rightarrow C = -2(2-9) = 14$$

Homework #8 #9

(9) cont'd...

Sin(t) - terms: $-\left(\frac{D}{2} + \frac{3E}{2}\right) = \frac{3}{2}$

$\Rightarrow D + 3E = -3$

cos(t) - terms: $-\frac{E}{2} + \frac{3D}{2} = 0$

$\Rightarrow -\frac{E}{2} = -\frac{3D}{2} \Rightarrow \underline{E = 3D}$

So: $D + 3E = D + 9D = 10D = -3$

$\Rightarrow D = -\frac{3}{10}$

$E = 3D = -\frac{9}{10}$

$$y_p(t) = t^2 - 6t + 14 - \frac{3}{10}\sin(t) - \frac{9}{10}\cos(t)$$

$$\therefore \boxed{y(t) = y_H(t) + y_p(t) = c_1 e^{-t/2} + c_2 e^{-t} + t^2 - 6t + 14 - \frac{3}{10}\sin(t) - \frac{9}{10}\cos(t)}$$

#10

(9) $u'' + \omega_0^2 u = \cos(\omega t) \quad (\omega^2 \neq \omega_0^2)$

Homogeneous piece:

$u'' + \omega_0^2 u = 0$

Let: $u = e^{rt}$; $r^2 + \omega_0^2 = 0 \Rightarrow r = \pm i\omega_0$

$u_H(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$

Particular piece:

$g(t) = \cos(\omega t), \quad \omega^2 \neq \omega_0^2$

Assume: $u_p(t) = A\sin(\omega t) + B\cos(\omega t)$

$u_p'(t) = A\omega\cos(\omega t) - B\omega\sin(\omega t)$

$u_p''(t) = -A\omega^2\sin(\omega t) - B\omega^2\cos(\omega t)$

Now: $u_p'' + \omega_0^2 u_p = \cos(\omega t)$

$-A\omega^2\sin(\omega t) - B\omega^2\cos(\omega t)$

$+\omega_0^2 A\sin(\omega t) + \omega_0^2 B\cos(\omega t) = \cos(\omega t)$

Homework #8

#11

(9) cont'd...

$$A(\omega_0^2-\omega^2)\sin(\omega t) + B(\omega_0^2-\omega^2)\cos(\omega t) = \cos(\omega t)$$

$$\Rightarrow A(\omega_0^2-\omega^2) = 0 \text{ and } B(\omega_0^2-\omega^2) = 1$$

Since $\omega_0^2 \neq \omega^2$, then $A = 0$.

$$B = \frac{1}{\omega_0^2-\omega^2}$$

So: $u_p(t) = \left(\frac{1}{\omega_0^2-\omega^2}\right)\cos(\omega t)$

$$\therefore \boxed{\begin{aligned} u(t) &= u_H(t) + u_p(t) \\ &\text{— or —} \\ u(t) &= c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{\cos(\omega t)}{\omega_0^2-\omega^2} \quad (\omega^2 \neq \omega_0^2) \end{aligned}}$$

[redacted]

17 $y'' + 4y = 3\sin(2t)$, $y(0) = 2$, $y'(0) = -1$

Homogeneous piece:

$$y'' + 4y = 0$$

Let: $y = e^{rt}$; $r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$$y_H(t) = c_1 \cos(2t) + c_2 \sin(2t)$$

Particular piece:

$$g(t) = 3\sin(2t)$$

Since $\sin(2t)$ is part of $y_H(t)$, we take: $y_p(t) = At\sin(2t) + Bt\cos(2t)$

$$y_p'(t) = A\sin(2t) + 2At\cos(2t) + B\cos(2t) - 2Bt\sin(2t)$$

Homework #8

17 cont'd...

$$y_p''(t) = 4A\cos(2t) - 4At\sin(2t) - 4B\sin(2t) - 4Bt\cos(2t)$$

<u>Now:</u> $y_p'' + 4y_p = 3\sin(2t)$

$$4A\cos(2t) - \cancel{4At\sin(2t)} - 4B\sin(2t) - \cancel{4Bt\cos(2t)} + \cancel{4At\sin(2t)} + \cancel{4Bt\cos(2t)} = 3\sin(2t)$$

$$\Rightarrow 4A\cos(2t) - 4B\sin(2t) = 3\sin(2t)$$

$$\Rightarrow 4A = 0 \ \underline{\text{and}} \ -4B = 3$$

$$\Rightarrow A = 0 \ \underline{\text{and}} \ B = -3/4$$

<u>So:</u> $y_p(t) = -\frac{3t}{4}\cos(2t)$

General solution:

$$y(t) = y_H(t) + y_p(t)$$

— or —

$$y(t) = c_1 \cos(2t) + c_2 \sin(2t) - \frac{3t}{4}\cos(2t)$$

#14

Note:

$$y'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t) - \frac{3}{4}\cos(2t) + \frac{3t}{2}\sin(2t)$$

$y(0)=2$: $c_1 = 2$

$y'(0)=-1$: $2c_2 - \frac{3}{4} = -1 \Rightarrow 2c_2 = -1 + \frac{3}{4} = \frac{-1}{4}$

$\Rightarrow c_2 = -1/8$

$$\therefore \boxed{y(t) = 2\cos(2t) - \frac{1}{8}\sin(2t) - \frac{3t}{4}\cos(2t)}$$

Homework #8

#1

Sec. 3.7: 4, 7, 9, 12, 17

(4) $4y'' - 4y' + y = 16e^{t/2}$

$$y'' - y' + \frac{1}{4}y = 4e^{t/2}$$ (Coeffient of y'' term must be $\underline{1}$!)

Homogeneous piece:

$$y'' - y' + \frac{1}{4}y = 0$$

Let: $y = e^{rt}$; $r^2 - r + \frac{1}{4} = 0$

$$r^2 - r + \frac{1}{4} = (r - 1/2)^2 = 0 \Rightarrow \underline{r = \tfrac{1}{2}}$$

So:

$$y_H(t) = c_1 e^{t/2} + c_2 t e^{t/2}$$

Particular piece:

(a) Use: Variation of parameters

Recall: $y_1(t) = e^{t/2}$, $y_2(t) = te^{t/2}$
(2 linearly independent solutions)

[redacted]

#2

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$y_1(t) = e^{t/2}, \quad y_1'(t) = \frac{1}{2} e^{t/2}$$

$$y_2(t) = t e^{t/2}, \quad y_2'(t) = e^{t/2} + \frac{1}{2} t e^{t/2}$$

<u>So:</u>

$$W(y_1, y_2) = e^{t/2}\left(e^{t/2} + \frac{1}{2} t e^{t/2}\right) - t e^{t/2}\left(\frac{1}{2} e^{t/2}\right)$$

$$= e^t + \frac{t}{2} e^t - \frac{t}{2} e^t = \underline{\underline{e^t}}$$

<u>Now:</u>

$$y_p(t) = -y_1(t) \int \frac{y_2(t)\, g(t)}{W(t)}\, dt + y_2(t) \int \frac{y_1(t)\, g(t)}{W(t)}\, dt$$

(4) cont'd...

Note: $g(t) = 4e^{t/2}$

$$y_p(t) = -e^{t/2}\int \frac{(te^{t/2})(4e^{t/2})}{e^t}\,dt$$

$$+\ te^{t/2}\int \frac{(e^{t/2})(4e^{t/2})}{e^t}\,dt$$

$$= -e^{t/2}\int 4t\,dt + te^{t/2}\int 4\,dt$$

$$= -2t^2e^{+t/2} + 4t^2e^{t/2}$$

$$= 2t^2e^{+t/2}$$

General solution:

$$y(t) = y_H(t) + y_p(t)$$

— or —

$$\boxed{y(t) = c_1e^{t/2} + c_2te^{t/2} + 2t^2e^{t/2}}$$

#4

(4) cont'd...

(b) Use: Method of undetermined coefficients

Recall: $y_H(t) = c_1 e^{t/2} + c_2 t e^{t/2}$

$$g(t) = 4e^{t/2}$$

Since $e^{t/2}$ and $te^{t/2}$ are part of the homogeneous solution, we must consider:

$$y_p(t) = At^2 e^{t/2}$$

$$y_p'(t) = 2Ate^{t/2} + \frac{1}{2}At^2e^{t/2}$$

$$y_p''(t) = 2Ae^{t/2} + 2Ate^{t/2} + \frac{1}{4}At^2e^{t/2}$$

Now: $y_p'' - y_p' + \frac{1}{4}y_p = 4e^{t/2}$

$$2Ae^{t/2} + \cancel{2Ate^{t/2}} + \frac{1}{4}At^2e^{t/2}$$
$$- \cancel{2Ate^{t/2}} - \frac{1}{2}At^2e^{t/2} + \frac{1}{4}At^2e^{t/2}$$
$$= 4e^{t/2}$$