§5.1: #15 Sol. To find all the eigenvectors, solve \((A - \lambda I)\vec{x} = \vec{0}\) where

\[
A - \lambda I = A - 3I = \begin{bmatrix}
1 & 2 & 3 \\
-1 & -2 & -3 \\
2 & 4 & 6
\end{bmatrix}.
\]

Reduce the augmented matrix to reduced echelon form

\[
\begin{bmatrix}
1 & 2 & 3 & | & 0 \\
-1 & -2 & -3 & | & 0 \\
2 & 4 & 6 & | & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 & 3 & | & 0 \\
0 & 0 & 0 & | & 0 \\
0 & 0 & 0 & | & 0
\end{bmatrix}.
\]

The linear system corresponding to the reduced echelon form is

\[
x_1 + 2x_2 + 3x_3 = 0 \\
0 = 0.
\]

The general solution is

\[
x_1 = -2x_2 - 3x_3 \\
x_2, x_3 \text{ free.}
\]

In vector form, it is

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
-2x_2 - 3x_3 \\
x_2 \\
x_3
\end{bmatrix} = x_2 \begin{bmatrix}
-2 \\
1 \\
0
\end{bmatrix} + x_3 \begin{bmatrix}
-3 \\
0 \\
1
\end{bmatrix}
\]

which gives all the eigenvectors associated with the eigenvalue \(\lambda = 3\) when \(x_2 \neq 0\) or \(x_3 \neq 0\). According to the definition, eigenspace is the set of all eigenvectors plus \(\vec{0}\), namely, \(\text{Null}(A - \lambda I) = \text{Null}(A - 3I)\). A basis for \(\text{Null}(A - 3I)\) is

\[
\{ \begin{bmatrix}
-2 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
-3 \\
0 \\
1
\end{bmatrix} \}.
\]

§5.1: #21 Sol.

(a) False. If \(A\vec{x} = \lambda\vec{x}\) for some \(\vec{x} \neq \vec{0}\), then \(\lambda\) is an eigenvalue of \(A\).

(b) True. If \(A\) is not invertible, then \(A\vec{v} = \vec{0} = c\vec{v}\) for some \(c \neq 0\) by (d), Theorem 8, §2.3. This implies that \(0\) is an eigenvalue of \(A\). Reversely, if \(0\) is an eigenvalue of \(A\), then \(A\vec{v} = \vec{0} = c\vec{v}\) for some \(c \neq 0\) by the definition of eigenvalues and eigenvectors. Thus the equation \(A\vec{x} = \vec{0}\) has a nontrivial solution \(\vec{v}\) and therefore \(A\) is not invertible by (d), Theorem 8, §2.3.

(c) True. If \(c\) is an eigenvalue of \(A\), then \(A\vec{v} = c\vec{v}\) for some \(\vec{v} \neq \vec{0}\) by the definition of eigenvalues and eigenvectors. This implies that \((A - cI)\vec{v} = \vec{0}\), namely, the equation \((A - cI)\vec{x} = \vec{0}\) has a nontrivial solution \(\vec{x} = \vec{v}\). Reversely, if the equation \((A - cI)\vec{x} = \vec{0}\) has a nontrivial solution \(\vec{x} = \vec{v}\), then \((A - cI)\vec{v} = \vec{0}\), and from which we have \(A\vec{v} = c\vec{v}\). So \(c\) is an eigenvalue of \(A\).
(d) True. To find an eigenvector associated with an eigenvalue $\lambda$, one needs to solve the linear system $Ax = \lambda x$. However, to check whether a given vector $v$ is an eigenvector, one just needs to compute and compare $Av$ and $\lambda v$.

(e) False. To find eigenvalues of $A$, one needs to solve the characteristic equation $\det(\lambda I - A) = 0$.

§5.2: #11 Sol. The characteristic polynomial of $A$ is
\[
\det(A - \lambda I) = \begin{vmatrix} 4 - \lambda & 0 & 0 \\ 5 & 3 - \lambda & 2 \\ -2 & 0 & 2 - \lambda \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}
\]
\[
= (4 - \lambda) \begin{vmatrix} 3 - \lambda & 2 \\ 0 & 2 - \lambda \end{vmatrix}
\]
\[
= (4 - \lambda)(3 - \lambda)(2 - \lambda)
\]

§5.2: #20 Sol. The characteristic polynomials of $A$ and $A^T$ are equal because
\[
\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I^T) = \det(A^T - \lambda I).
\]
The first equation above follows from Theorem 5, §3.2.