ON A SUBSPACE RELATED TO THE KOROVKIN CLOSURE [1]

Si-Kit Chung(†) and Man-Chung Yeung(‡)

(†) Department of Mathematics, University of Illinois, Urbana, IL 61801.
(‡) Department of Mathematics, University of California, Los Angeles, CA 90024.

Abstract

Let L be a closed subspace of $\mathcal{C}(X)$ which separates points and contains the constants. Denote the Korovkin closure of L by $\widehat{L}$. Then $L = \widehat{L} \cap M_L$ where $M_L = \{f \in \mathcal{C}(X) : \int f \; d(\mu \circ j) = 0$ for all boundary dependences μ on $K_L\}$. We consider the relation between L and M_L, the Choquet boundary of M_L and the state space of M_L.

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[1] Dedicated to the late Professor A.J. Ellis

1 Introduction

Let X be a compact Hausdorff space and $\mathcal{C}(X)$ be the space of all continuous real-valued functions on X with the sup norm. Let L be a closed subspace of $\mathcal{C}(X)$ which contains the constant functions and separates points on X. The Korovkin closure of L, denoted by $\widehat{L}$, has been studied by Bauer in [2, 3, 4]. Let K_L be the state space of L and $j : X \to K_L$ the canonical injection. Here we consider a closed subspace M_L of $\mathcal{C}(X)$ defined by

$$M_L = \{f \in \mathcal{C}(X) : \int f \; d(\mu \circ j) = 0 \text{ for all boundary dependences } \mu \text{ on } K_L\} \; .$$

In [4], Bauer et al. introduced the idea of psuedo-affine functions on a compact subset of K_L (with the induced topology of the weak*-topology on L^*) and has obtained a characterization of $\widehat{L}$. By [4, Proposition 1], it is not difficult to see that $L = \widehat{L} \cap M_L$ when $\overline{Ch(L)} = X$. In this paper, we will see that $L = \widehat{L} \cap M_L$ holds in general and then we will consider the relation between L and M_L, the Choquet boundary of M_L and the state space of M_L.

2 Relation Between L and M_L

To study the problem mentioned above, we consider $\widetilde{L} = \{f \in \mathcal{C}(X) : f(x) = 0 \text{ for all } x \in Ch(L)\}$ where $Ch(L)$ is the Choquet boundary of L.

Proposition 1 *$\widehat{L} \cap \widetilde{L} = \{0\}$.*

Proof Let $f \in \widehat{L}$ be such that $f(x) = 0$ for all $x \in Ch(L)$. Since the Choquet boundary for $\widehat{L}$ is a boundary for $\widehat{L}$ and $Ch(\widehat{L}) = Ch(L)$, so we have $f(x) = 0$ for all $x \in X$. ■

Corollary 1 *$\widehat{L} \cap M_L = L$.*

Proof $\widehat{L} \cap M_L \supset L$ is clear. Next if $f \in \widehat{L} \cap M_L$ then, by [4, Proposition 1], we see that there is a $g \in L$ such that $g|_{\overline{Ch(L)}} = f|_{\overline{Ch(L)}}$. So $f - g \in \widehat{L} \cap \widetilde{L}$ which is $\{0\}$ by Proposition 1, and hence $f = g \in L$. ■

Proposition 2 *$L \oplus \widetilde{L} \subset M_L$.*

Proof Since $L \subset \widehat{L}$ we have, by Proposition 1, $L \cap \widetilde{L} = \{0\}$. Next we want to show that $L + \widetilde{L}$ is closed. Suppose $f_n + g_n$ converges uniformly on X to h where $f_n \in L$, $g_n \in \widetilde{L}$ and $h \in \mathcal{C}(X)$. Since

$$\begin{aligned} \|f_n - f_m\|_\infty &= \sup_{x \in Ch(L)} |(f_n - f_m)(x)| \\ &= \sup_{x \in Ch(L)} |(f_n + g_n)(x) - (f_m + g_m)(x)| \\ &\leq \|(f_n + g_n) - (f_m + g_m)\|_\infty \to 0 \quad \text{as } m, n \to \infty \; , \end{aligned}$$

so f_n converges uniformly to an $f \in L$, hence g_n converges uniformly to $(h - f) \in \widetilde{L}$. Therefore $h = f + (h - f) \in L + \widetilde{L}$. To show $L + \widetilde{L} \subset M_L$, it suffices to show $\widetilde{L} \subset M_L$ since $L \subset M_L$. However, this is obvious since maximal measures and hence boundary dependences on K_L are supported by ext $K_L = j(Ch(L))$. ■

Remarks (a) Using the same argument as in Proposition 2, we see that $\widehat{L} + \widetilde{L}$ is closed also.

(b) The inclusion in Proposition 2 may be proper as the following example illustrates.

Example 1 Let $X = [0,1]$ and $L = \{f \in \mathcal{C}(X) : f(0) = \int_0^1 f\ dm\}$ where m is the Lebesgue measure on [0, 1]. Clearly L is a closed subspace of $\mathcal{C}(X)$, separating points on X and containing the constants. It is clear that $m \underset{L}{\sim} \varepsilon_0$, so $0 \notin Ch(L)$. Moreover, for any $x \in [0,1] \setminus \{0\}$, there is an f in L with x as the peak point. Hence $Ch(L) = [0,1] \setminus \{0\}$. Therefore $\widetilde{L} = \{0\}$. Next we want to show that any boundary dependence must be the zero measure so that $M_L = \mathcal{C}[0,1] \underset{\neq}{\supset} L + \widetilde{L} = L$. Let $\mu = \mu_1 - \mu_2$ where μ_1, μ_2 are maximal measures on K_L with $r(\mu_1) = r(\mu_2)$. Since $\mu \circ j \in M_L^{\perp}$ and $L \subset M_L$, we have $\mu \circ j \in L^{\perp} = \text{span}\ \{\varepsilon_0 - m\}$. Let $\mu \circ j = \alpha(\varepsilon_0 - m)$. Then we have $\mu(j(\{0\})) = \alpha$. But $\mu_1(j(\{0\})) = \mu_2(j(\{0\})) = 0$ since maximal measures on K_L are supported by ext $K_L = j(Ch(L)) = j((0,1])$. Hence $\alpha = 0$ and so we have $\mu = 0$.

As we have seen in the above example that the inclusion in Proposition 2 may be improper, so we would like to find some conditions that imply $L + \widetilde{L} = M_L$ holds.

Proposition 3 *If K_L is stable, then $L + \widetilde{L} = M_L$.*

Proof For any $f \in M_L$, define a function F on K_L by $F(z) = \int f\ d(\mu_z \circ j)$ where μ_z is any maximal measure on K_L with $r(\mu_z) = z$. The definition of $F(z)$ does not depend on the choice of μ_z since $f \in M_L$. Moreover F is affine on K_L. Indeed, for any z_1, $z_2 \in K_L$ and $0 \leq t \leq 1$, let $\mu_i (i = 1, 2)$ be maximal measures on K_L such that $r(\mu_i) = z_i$. Then $t\mu_1 + (1-t)\mu_2$ is a maximal measure with $r(t\mu_1 + (1-t)\mu_2) = tz_1 + (1-t)z_2$. Hence

$$\begin{aligned} F(tz_1 + (1-t)z_2) &= \int f\ d((t\mu_1 + (1-t)\mu_2) \circ j) \\ &= t \int\ f\ d(\mu_1 \circ j) + (1-t) \int\ f\ d(\mu_2 \circ j) \\ &= t\ F(z_1) + (1-t)\ F(z_2)\ . \end{aligned}$$

Next we want to show that F is continuous on K_L. Let $\{z_\alpha\}$ be a net in K_L which converges to z_0 in K_L. Since K_L is stable, the resultant map restricted to the set of all maximal measures is an open map (cf. [6, Corollary 2]). Therefore we can choose a maximal measure μ_0 with $r(\mu_0) = z_0$ and a net of maximal measures $\{\mu_\beta\}$ converging to μ_0 in the weak*-topology of $\mathcal{C}(K_L)^*$ such that $\{r(\mu_\beta) = z_{\alpha_\beta}\}$ is a subnet of $\{z_\alpha\}$. Therefore

$$F(z_0) = \int\ f\ d(\mu_0 \circ j) = \lim_\beta \int\ f\ d(\mu_\beta \circ j) = \lim_\beta F(z_{\alpha_\beta})\ .$$

¿From this we see that F is continuous and so $F \in \mathcal{A}(K_L)$ where $\mathcal{A}(K_L)$ is the set of all continuous affine functions on K_L. Therefore, there is a $g \in L$ such that $F \circ j = g$. However, if $x \in Ch(L)$, we have $j(x) \in$ ext K_L, so that $\mu_{j(x)}$ must be $\varepsilon_{j(x)}$, and hence $g(x) = F(j(x)) = \int\ f\ d(\varepsilon_{j(x)} \circ j) = f(x)$. Therefore, $f = g + (f - g)$ belongs to $L + \widetilde{L}$. ■

Remark In the above proposition, the condition that K_L is stable can be replaced by a weaker condition: "ext K_L is closed". This can be seen as a direct consequence of Proposition 5 and Theorem 1 and so we omit the proof here. However, this weaker condition is only sufficient but not necessary as the following example illustrates.

Example 2 Let $X = \{(x, y, z) \in \mathbf{R}^3 : (x-1)^2 + y^2 = 1 \text{ and } z = 0\} \cup \{(0,0,1), (0,0,-1)\}$ and $L = \{f \in \mathcal{C}(X) : f = F|_X \text{ for some } F \in \mathcal{A}(\text{conv } X)\}$ where conv X is the convex hull of X. Then it is not difficult to see that $K_L = \text{conv } X$ and that ext $K_L = Ch(L) = X \setminus \{(0,0,0)\}$ which is not closed. However by direct verification, or by the next proposition, we have $M_L = L = L + \widetilde{L}$.

Proposition 4 *If K_L is finite-dimensional, then $L + \widetilde{L} = M_L$.*

Proof We can apply the proof of Proposition 3 and note that the function F defined is continuous if K_L is finite-dimensional. ■

Corollary 2 *If $Ch(L) = X$, then $L = M_L$.*

Proof If $Ch(L) = X$, then $\widetilde{L} = \{0\}$ and ext $K_L = j(X)$ is closed. So by the remark following Proposition 3, we have $M_L = L + \widetilde{L} = L$. ■

Remark There is a simple proof for the above corollary. Indeed, $Ch(L) = X$ implies $\widehat{L} = \mathcal{C}(X)$, hence $M_L = L$ by Corollary 1.

3 Another Approach

In [4], Bauer introduced the idea of psuedo-affine functions on a compact subset of a compact convex set in a locally convex space. We will use this idea to give a necessary and sufficient condition on K_L such that $L + \widetilde{L} = M_L$ holds. Before this, we consider the following

Definition Let K be a compact convex set in a locally convex space. A continuous function F on $\overline{\text{ext } K}$ is said to be a *boundary function* if $\int F \, d\mu = 0$ for all boundary dependences on K. K is said to be *psuedo-stable* if every boundary function is psuedo-affine on $\overline{\text{ext } K}$.

Note If $K = K_L$ is given by some L, then $f \circ j^{-1}|_{\overline{\text{ext } K_L}}$ are boundary functions for every $f \in M_L$. Also any $f \in M_L$ is a continuous extension of $F \circ j|_{\overline{Ch(L)}}$ for some boundary function F.

Before proving the main result of this section, we give some examples of psuedo-stable compact convex sets in the following two propositions.

Proposition 5 *If* ext *K is closed, then K is psuedo-stable. In particular, K is stable implies K is psuedo-stable.*

Proof If ext K is closed, then any continuous function on ext K is pseudo-affine on ext K, so K is psuedo-stable. If K is stable, then ext K is closed by [9, Corollary 3.1]. ■

Proposition 6 *If K is finite-dimensional, then K is psuedo-stable.*

Proof Let $F \in \mathcal{C}(\overline{\text{ext } K})$ be a boundary function. Since K is finite-dimensional, $K = \text{conv}\,(\text{ext } K)$. So for any $z \in K$, there exist $z_1, \cdots, z_n \in \text{ext } K$ and $t_i \geq 0$ $(i = 1, \cdots, n)$ with $\sum_{i=1}^n t_i = 1$ such that $z = \sum_{i=1}^n t_i z_i$. Define $G(z) = \sum_{i=1}^n t_i \, F(z_i)$. This is well-defined since F is a boundary function and $\sum_{i=1}^n t_i \varepsilon_{z_i}$ is a maximal measure with resultant z. From the definition of G, it is clear that $G \in \mathcal{A}(K)$ and so $F = G|_{\overline{\text{ext } K}}$ is psuedo-affine on $\overline{\text{ext } K}$. Hence K is psuedo-stable. ■

Theorem 1 *$L + \widetilde{L} = M_L$ if and only if K_L is psuedo-stable.*

Proof ($\Longrightarrow$) Let $F \in \mathcal{C}(\overline{\text{ext } K_L})$ be a boundary function. Choose an $f \in \mathcal{C}(X)$ which extends $F \circ j|_{\overline{Ch(L)}}$. Clearly f belongs to M_L. Let $f = g + h$ where $g \in L$ and $h \in \widetilde{L}$. Then we have $f(x) = g(x)$ for all $x \in \overline{Ch(L)}$. But there is a $G \in \mathcal{A}(K_L)$ such that $G \circ j = g$. Hence $F = G|_{\overline{\text{ext } K_L}}$ is psuedo-affine on $\overline{\text{ext } K_L}$ and so K_L is psuedo-stable.

($\Longleftarrow$) Let $f \in M_L$. Then $F = f \circ j^{-1}|_{\overline{\text{ext } K_L}}$ is a boundary function. So K_L is psuedo-stable implies F is psuedo-affine on $\overline{\text{ext } K_L}$. By [4, Proposition 1], there is a $G \in \mathcal{A}(K_L)$ such that $F = G|_{\overline{\text{ext } K_L}}$. Let $g \in L$ satisfies $G \circ j = g$. Then we have $f(x) = g(x)$ for all $x \in Ch(L)$ and so $f = g + (f - g)$ where $(f - g) \in \widetilde{L}$. Therefore $M_L = L + \widetilde{L}$ holds. ■

¿From the above theorem, we immediately have the following

Corollary 3 *$L = M_L$ if and only if $\overline{Ch(L)} = X$ and K_L is psuedo-stable.*

¿From the discussion above and that in the previous section, we may ask whether K is psuedo-stable implies ext K is closed or K is finite-dimensional. The answer is negative.

Example 3 Let $X_1 = X \cup \{(x, y, z) : y = z = 0,\ 3 \leq x \leq 4\}$ where X is the compact set given in Example 2. Using the subspace L of $\mathcal{C}(X)$ in Example 2, we construct $L_1 = \{f \in \mathcal{C}(X_1) : f|_X \in L\}$. Then we have $Ch(L_1) = Ch(L) \cup \{(x, 0, 0) : 3 \leq x \leq 4\}$. So ext K_{L_1} is not closed. Clearly K_{L_1} is infinite-dimensional. Moreover K_{L_1} is psuedo-stable since $M_{L_1} = L_1$.

Before going to the next section, we give the following example as an application of the above result in determining M_L.

Example 4 Let $X = [0, 1]$ and $L = \text{span}\{1, id, u\}$ where id is the identity function and u is a continuous function on [0, 1].

When u is given by $u(t) = \begin{cases} c_1 & \text{if } t \in [0, \alpha] \\ \text{linear on } [\alpha, \beta] & \\ c_2 & \text{if } t \in [\beta, 1] \end{cases}$ where $c_1 \neq c_2$ and $0 < \alpha < \beta < 1$.

Then we have

$$M_L = \{g \in \mathcal{C}[0,1] : \frac{g(\alpha) - g(0)}{\alpha} = \frac{g(1) - g(\beta)}{1 - \beta}\}$$

since $M_L = L + \widetilde{L}$ and

$$L = \{f \in \mathcal{C}[0,1] : f \text{ is linear on } [0, \alpha], [\alpha, \beta] \text{ and } [\beta, 1] \text{ and } \frac{f(\alpha) - f(0)}{\alpha} = \frac{f(1) - f(\beta)}{1 - \beta}\}.$$

When u is a concave (or convex) function which is piecewise linear on n sub-intervals of [0, 1], then by [4, Example following Proposition 7], we have

$$\widehat{L} = \{f \in \mathcal{C}[0,1] : f \text{ is linear on the } n \text{ sub-intervals given}\} .$$

Hence $\widehat{L} + \widetilde{L} = \mathcal{C}[0,1]$ and $\dim \widehat{L} = n + 2$, so that $M_L = L + \widetilde{L}$ has co-dimension $n - 2$.

In the particular case when $n = 2$ in (b), M_L has co-dimension 0, that is, $M_L = \mathcal{C}[0,1]$. Note that in this case K_L is a triangle which is a simplex and K_{M_L} is a simplex too. We will prove a general result related to this later.

4 Some Properties on the Choquet Boundary of M_L

Since L is contained in M_L, we have $Ch(L) \subset Ch(M_L)$. Moreover, $Ch(M_L)$ is a 'large' set, as the following result shows.

Proposition 7 *$Ch(M_L)$ is dense in X.*

Proof Let V be a non-empty open set. We want to show that $Ch(M_L) \cap V \neq \phi$. We may assume $V \cap Ch(L) = \phi$. Let $Y = X \setminus V$. Take any non-zero $f \in \mathcal{C}(X)$ with $f|_Y = 0$. Then f belongs to M_L. Since $Ch(M_L)$ is a boundary for M_L, there is an $x_0 \in Ch(M_L)$ such that $|f(x_0)| \geq |f(x)|$ for all $x \in X$. But the condition on f implies that $x_0 \in V$. Hence $Ch(M_L)$ intersects V. ■

Corollary 4 *If $L = M_L$ and $Ch(M_L)$ is closed, then $\widehat{L} = \mathcal{C}(X)$.*

We may have $Ch(L)$ a proper subset of $Ch(M_L)$ and $Ch(M_L)$ a proper subset of X.

Example 5 Let $X_2 = X \cup \{(x, 0, 0) : 3 \leq x \leq 5\}$ where X is the compact set given in Example 2. Let $L_2 = \{f \in \mathcal{C}(X_2) : f|_X \in L$ and f is linear on $\{(x, 0, 0) : 3 \leq x \leq 4\}$ and on $\{(x, 0, 0) : 4 \leq x \leq 5\}\}$. Then we have $Ch(L_2) = Ch(L) \cup \{(3, 0, 0), (4, 0, 0), (5, 0, 0)\}$. Since $M_{L_2} = \{f \in \mathcal{C}(X_2) : f|_X \in L\}$, we obtain $Ch(M_{L_2}) = Ch(L) \cup \{(x, 0, 0) : 3 \leq x \leq 5\}$. Hence we have $Ch(L_2) \underset{\neq}{\subset} Ch(M_{L_2}) \underset{\neq}{\subset} X_2$.

¿From the above discussion, we may ask when do L and M_L have the same Choquet boundary? Clearly when $L = M_L$, this is true. Is this the only possible case? The next two results answer this question partially.

Proposition 8 *When K_L is psuedo-stable, we have*

$$Ch(L) = Ch(M_L) \text{ if and only if } L = M_L\ .$$

Proof Suppose that $Ch(L) = Ch(M_L)$. Then by Proposition 7, $\overline{Ch(L)} = \overline{Ch(M_L)} = X$. So when K_L is psuedo-stable, we have $L = M_L$ by Corollary 3. ■

Before proving the next result, we need to note the following fact. If μ is a maximal measure on K_L, then it is supported by $j(X)$ and so it can be considered as a measure on K_{M_L}. Similarly any maximal measure on K_{M_L} can be considered as a measure on K_L also. So, the condition in the lemma below makes sense.

Lemma 1 *If every maximal measure on K_{M_L} is also a maximal measure on K_L, then $M_L = L$.*

Proof Firstly, we note that the given condition implies $\overline{Ch(L)} = X$. If we suppose not then, by Proposition 7, there exists an $x_0 \in Ch(M_L)$ and $x_0 \notin \overline{Ch(L)}$. So $\varepsilon_{j(x_0)}$ is a maximal measure on K_{M_L} but not maximal on K_L.

Next, we consider the function F on K_L defined in Proposition 3. Note that once we have proved that F is continuous, the remaining argument in Proposition 3 still works and so together with the fact that $\overline{Ch(L)} = X$, we have $M_L = L$.

Let $z_0 \in K_L$ and $\{z_\alpha\}$ be a net in K_L converging to z_0. Choose a net of maximal measures $\{\mu_\alpha\}$ on K_L with $r(\mu_\alpha) = z_\alpha$. Since the unit ball of $\mathcal{C}(K_L)^*$ is weak*-compact, there is a subnet $\{\mu_{\alpha_\beta}\}$ of $\{\mu_\alpha\}$ which converges to μ_1 in the weak*-topology. Since the resultant map is continuous, we have $r(\mu_1) = z_0$. This μ_1 is a probability measure but not necessary maximal on K_L. However, μ_1

is supported by $j(X)$ and so we can choose a maximal measure μ_2 on K_{M_L} such that $\mu_2 \underset{K_{M_L}}{\succ} \mu_1$. ¿From the fact that $L \subset M_L$, we have $r(\mu_2) = z_0$. Hence we obtain

$$F(z_0) = \int f \; d(\mu_2 \circ j) = \int f \; d(\mu_1 \circ j) = \lim_{\beta} \int f \; d(\mu_{\alpha_\beta} \circ j) = \lim_{\beta} F(z_{\alpha_\beta}) \; .$$

¿From this we see that F is continuous and the proof is completed. ■

Theorem 2 *If X is metrizable, then we have*

$$Ch(L) = Ch(M_L) \text{ if and only if } L = M_L \; .$$

Proof Suppose that X is metrizable and let μ be a maximal measure on K_{M_L}. Then μ is supported by $j(Ch(M_L))$ and so μ is maximal on K_L if $Ch(L) = Ch(M_L)$. Hence the result follows from Lemma 1. ■

Remark We do not know whether $Ch(M_L) = Ch(L) \iff M_L = L$ holds in general or not.

5 Some Properties of the State Space of M_L

In this section, we will show that the state space of M_L is always psuedo-stable and that it cannot be a simplex unless M_L is the whole space $\mathcal{C}(X)$.

Proposition 9 $M_{M_L} = M_L$.

Proof To prove this, we need the following fact:
"Let L_1 and L_2 be closed subspaces of $\mathcal{C}(X)$ containing the constants and separating points. If $L_1 \subset L_2$, then all maximal measures on K_{L_1} are also maximal measures on K_{L_2}."

Now, $M_L \subset M_{M_L}$ is clear. Next, let $f \in M_{M_L}$, we want to show that $f \in M_L$. Let $\mu = \mu_1 - \mu_2$ be any boundary dependence on K_L, that is, μ_1, μ_2 are maximal measures on K_L with the same resultant. By the above fact, μ_1, μ_2 are also maximal measures on K_{M_L} since $L \subset M_L$. Moreover, for any $g \in M_L$, we have $\int g \; d(\mu \circ j) = 0$ or $\int g \; d(\mu_1 \circ j) = \int g \; d(\mu_2 \circ j)$. So μ_1, μ_2 have the same resultant in K_{M_L}, and hence μ is a boundary dependence on K_{M_L}. Therefore we have $\int f \; d(\mu \circ j) = 0$ since $f \in M_{M_L}$. Hence f belongs to M_L by definition. ■

Corollary 5
(a) *K_{M_L} is psuedo-stable.*
(b) *$\overline{Ch(M_L)} = X$.*

Proof This follows immediately from Proposition 9 and Corollary 3. In fact, part (b) has already been proved in Proposition 7. ■

Proposition 10 *The following assertions are equivalent.*
(a) *$M_L = \mathcal{C}(X)$.*
(b) *K_L is a simplex.*
(c) *K_{M_L} is a simplex.*

Proof (a) $\Longleftrightarrow$ (b) $M_L = \mathcal{C}(X)$ is equivalent to each boundary dependence on K_L being zero which in turn is equivalent to any point on K_L having only one maximal representing measure, that is, K_L is a simplex.
(a) $\Longrightarrow$ (c) This is obvious.
(c) $\Longrightarrow$ (a) If K_{M_L} is a simplex, then by the equivalence of (a) and (b), we have $M_{M_L} = \mathcal{C}(X)$, and hence $M_L = \mathcal{C}(X)$ by Proposition 9. ■

Corollary 6 *If K_{M_L} is a simplex, then $L = \widehat{L}$.*

We now return to an arbitrary compact convex set K in a locally convex space. In Section 2, we saw that ext K is closed implies that K is psuedo-stable, but that the converse is not true. However, when K is a simplex, these two concepts are the same.

Theorem 3 *Let K be a compact convex set in a locally convex space. Then the following are equivalent.*
(a) *K is a Bauer simplex.*
(b) *K is a psuedo-stable simplex.*
(c) $\mathcal{A}(K)|_{\overline{\text{ext } K}} = \mathcal{C}(\overline{\text{ext } K})$.

Proof (b) $\Longleftrightarrow$ (c) Consider the compact Hausdorff space $X = K$ and the closed subspace $L = \mathcal{A}(K)$ of $\mathcal{C}(X)$ which separates points and contains the constants. Then $Ch(L) =$ ext K and K_L can be identified as K. So by Theorem 1 and Proposition 10, the property that $K = K_L$ is a psuedo-stable simplex is equivalent to

$$\mathcal{C}(K) = \mathcal{C}(X) = M_L = L + \widetilde{L} = \mathcal{A}(K) + \widetilde{L}\ .$$

This in turn is equivalent to $\mathcal{A}(K)|_{\overline{\text{ext } K}} = \mathcal{C}(\overline{\text{ext } K})$.

The equivalence of (a) and (c) is well known (see [1]).

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