A completely integrable particle method for a nonlinear shallow-water wave equation in periodic domains

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Abstract. We propose an algorithm for an asymptotic model of shallow-water wave dynamics in a periodic domain. The algorithm is based on the Hamiltonian structure of the equation and corresponds to a completely integrable particle lattice. In particular, "periodic particles" are introduced in the algorithm for waves travelling through the domain. Each periodic particle in this method travels along a characteristic curve of the shallow-water wave model, determined by solving a system of nonlinear integral-differential equations. Accuracy tests for assessing the global properties of the method are performed and compared to more standard PDE algorithms.

Keywords. Shallow-water wave equation, intergrable system, particle method, periodic boundary conditions.

1 Introduction

The nonlinear partial differential equation (PDE) of evolution

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1}$$

results from an asymptotic expansion of the Euler equations governing the motion of an inviscid fluid whose free surface can exhibit gravity driven wave motion [5]. The small parameters used to carry out the expansion are the aspect ratio, whereby the depth of the fluid is assumed to be much smaller than the typical wavelength of the motion, and the amplitude ratio, or ratio between a typical amplitude of wave motion and the average depth of the fluid. Thus, the equation is a member of the class of weakly nonlinear (due to the smallness assumption on the amplitude parameter) and weakly dispersive (due to the long wave assumption parameter) models for water wave propagation. However, at variance with its celebrated close relatives in this class, such as the Korteweg-de Vries (KdV) and Benjamin-Bona-Mahony (BBM) equations, these small parameters are assumed to be linked only by a relative ordering, rather than a power law relation. This allows to retain terms on the right hand side that would be of higher order with respect to both the KdV and BBM expansions, and, in principle, consider dynamical regimes in which nonlinearity is somewhat dominant with respect to wave dispersion.

The above nonlinear equation possesses the remarkable property of complete integrability, as evidenced by its Lax-pair representation. Moreover, this property is complemented by the existence of a class of weak solutions that can serve as a natural projection of the general solution of (1) to an approximating (but still completely integrable) finite dimensional dynamical system [4,6,7]. This system of ordinary differential equations (ODEs) can be viewed as describing particle interacting through a long range potential (here position and momentum dependent), which expresses the fact that such particles are advected by the velocity u of the shallow water wave equation (1). The velocity is in turn determined by the particle positions and momenta. Previously such a particle system has been used to develop particle algorithms for solving the evolution equation (1) in the infinite domain, and the "quarter-plane problem" with zero boundary condition at the origin [6]. The present work focuses on developing and analyzing the particle algorithm in a periodic domain. We introduce "periodic particles" by summing all periods of the kernel in the system of integrable equations. Just as its infinite line counterpart, the emphasis of the present investigation is on simplicity and efficiency of the particle algorithm in periodic domains, rather than high-order accuracy, though this can be achieved with a little extra effort.

2 The integrable formulation and particle method

In this section we review briefly the particle algorithm developed in [4,6,7]. By introducing the characteristics $x = q(\xi, t)$,

$$\frac{dq}{dt} = u(q(\xi,t),t)\,, \qquad q(\xi,0) = \xi\,, \tag{2}$$

a solution of equation (1) in the infinite domain follows formally from the Hamiltonian system

$$\begin{aligned} q_t(\xi,t) &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|q(\xi,t)-q(\eta,t)|}p(\eta,t)d\eta - \kappa, \\ p_t(\xi,t) &= \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{sgn}(\xi-\eta)\, e^{-|q(\xi,t)-q(\eta,t)|}\, p(\xi,t)p(\eta,t)d\eta. \end{aligned} \tag{3}$$

Here the characteristics $q(\xi,t)$ play the role of positions conjugate to the momentum-like variables $p(\xi,t)$ [4] in the Hamiltonian

$$H = \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(e^{-|q(\xi,t)-q(\eta,t)|}\,p(\eta,t)\,p(\xi,t) - \kappa\big(p(\xi,t)+p(\eta,t)\big)\Big)d\eta d\xi\,,$$

which yields system (3) by the (standard) Poisson structure

$$q_t = \frac{\delta H}{\delta p}\,, \qquad p_t = -\frac{\delta H}{\delta q}\,,$$

where $\delta/\delta q$, $\delta/\delta p$ denote functional derivatives with respect to the functions $q(\xi,t)$ and $p(\xi,t)$, respectively, at fixed time t. The choice of initial condition for the position variable, dictated by the characteristics condition, implies $q_\xi(\xi,0)=1$, so that the constraint

$$q_\xi(\xi,t) = \frac{p(\xi,0)}{p(\xi,t)} \tag{4}$$

is maintained at all times of existence of the solution $(q(\xi,t),p(\xi,t))$. Thus, the momentum variable $p(\xi,t)$ could be eliminated from the system to obtain an evolution equation containing only the dependent variable $q(\xi,t)$ and its first derivative with respect to the initial label ξ. Vanishing of this derivative generically corresponds to crossing of characteristics curves, with loss of uniqueness of solutions $\xi(x)$ to the equation $x = q(\xi,\cdot)$. Constraint (4) then shows that if the initial condition $p(\xi,0)$ does not have zeros, $q_\xi(\cdot,t)$ is bounded away from zero, thereby preventing characteristics from crossing, for as long as $|p(\cdot,t)|<\infty$ [4]. The relation of system (3) with the original form (1) of the shallow water wave equation results from the definition of the velocity $u(x,t)$ in terms of characteristics $q(\xi,t)$ and the conjugate momentum $p(\xi,t)$,

$$u(x,t) = -\kappa + \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-q(\eta,t)|} p(\eta,t)\,d\eta\,. \tag{5}$$

System (3) can be rewritten in a slightly different form [6,7], and the numerical algorithm proposed in [4] approximates the integrals in this system by their Riemann sums, thereby yielding Hamiltonian systems for "particles" with coordinates

$$q_i(t) \equiv q(\xi_i,t)$$

and momenta

$$p_i(t) \equiv p(\xi_i,t),$$

where $\xi_i = \Xi + ih$ for some real Ξ, step-size $h>0$ and $i=1,\cdots,N$.

By replacing q_η in the system of integrals by the constraint (4), the discretized version of the system results in the finite dimensional system of ODEs of N particles,

$$\begin{aligned}
\dot{q_i} &= \frac{h}{2}\sum_{j=1}^{N} e^{-|q_i-q_j|}p_j - \frac{h}{2}\,\kappa\sum_{j=1}^{N} e^{-|q_i-q_j|}\,p_j^0/p_j,\\
\dot{p_i} &= \frac{h}{2}\,p_i\sum_{i\neq j=1}^{N}\operatorname{sgn}(q_i-q_j)e^{-|q_i-qj|}\,p_j - \\
&\quad \frac{h}{2}\,\kappa\,p_i\sum_{i\neq j=1}^{N}\operatorname{sgn}(q_i-q_j)e^{-|q_i-q_j|}\,p_j^0/p_j,
\end{aligned} \tag{6}$$

where $p(\xi_j,0)\equiv p_j^0$. System (6) constitutes our *particle method* for solving the shallow-water wave equation (1); it can be shown to provide a convergent numerical algorithm to solutions of (1)under appropriate assumptions on the initial data [6,7,8].

3 The particle method in periodic domains

Periodic domains are often used in numerical simulations for PDEs, since no actual numerical boundary conditions are required for their implementation. This is advantageous for complex systems, for which construction of numerical boundary conditions that are consistent with those dictated by physical considerations is often less than straightforward. Periodic domains also have the advantage of computational efficiency, since the computational domain can then be restricted to the window of a single period. While these advantages are clear for the Eulerian implementation of numerical algorithms, they can pose a challenge for Lagrangian methods, such as the particle method developed in [4,6,7], because particles can escape the period domain during their time evolution at times that cannot be known in advance. However, in the specific case of equation (1), a particle method in periodic domains can be implemented by observing that the periodic solutions $u(x,t)$ can be achieved by considering a periodic extension of the exponential kernel in (5) by superposition of period shifted exponentials (see also [1,2])

$$\phi_L(x,q) = \sum_{k=-\infty}^{\infty} e^{-|x-(q+kL)|}, \tag{7}$$

where L is the period. We will refer to equation (7) as *the periodic kernel.* Equation (7) implies that one can always impose $0\le |x-q|\le L$. We can express (7) in a more compact form by splitting the doubly infinite sum into its negative and positive index n range and looking at the two possible cases (1) $0\le x-q\le L$ and (2) $-L\le x-q\le 0$. For case (1), we have

$$\begin{aligned}
\phi_L(x,q) &= \sum_{k=-\infty}^{0} e^{-(x-q)+kL} + \sum_{k=1}^{\infty} e^{(x-q)-kL}\\
&= \frac{e^{-(x-q)}}{1-e^{-L}} + \frac{e^{x-q}e^{-L}}{1-e^{-L}}\\
&= \frac{\cosh(x-q-L/2)}{\sinh(L/2)}.
\end{aligned} \tag{8}$$

Similarly, for case (2), the periodic kernel is

$$\phi_L(x,q) = \frac{\cosh(x-q+L/2)}{\sinh(L/2)}. \tag{9}$$

Combining (8) and (9), we have

$$\phi_L(x,q) = \frac{\cosh\big(|(x-q)_{\bmod L}| - L/2\big)}{\sinh(L/2)}, \tag{10}$$

where $(x-q)_{\bmod L}$ is modulus of $x-q$ and L.

If we replace the kernel in the first equation of (3) by the periodic kernel (10) and integrate it over one period, we obtain an evolution equation for q

$$q_t(\xi,t) = \frac{1}{2}\int_{-L/2}^{L/2}\phi_L(q(\xi,t),q(\eta,t))\,p(\eta,t)\,d\eta - \kappa. \tag{11}$$

Taking the time derivative for the auxiliary function $p(\xi,t)$, as defined in [4] (equation (2.12) in that reference), and using (11), yields the evolution equation for p

$$p_t(\xi,t) = -\frac{1}{2}\,p(\xi,t)\int_{-L/2}^{L/2} \mathrm{sgn}(\xi-\eta)\times \psi_L(q(\xi,t),q(\eta,t))p(\eta,t)\,d\eta, \tag{12}$$

where

$$\psi_L(x,q) = \frac{\sinh\big(|(x-q)_{\mathrm{mod}\,L}| - L/2\big)}{\sinh(L/2)}. \tag{13}$$

Evaluating the integrals in (11) and (12) by the trapezoidal rule results in the finite dimensional system of ODEs of N particles in a periodic domain

$$\begin{aligned}\dot{q_i} &= \frac{h}{2}\sum_{j=-N/2}^{N/2-1}\phi_L(q_i,q_j)p_j - \kappa\\ \dot{p_i} &= -\frac{h}{2}\,p_i\sum_{i\neq j=-N/2}^{N/2-1}\psi_L(q_i,q_j)p_j.\end{aligned} \tag{14}$$

It is worth pointing out that since it is not necessary to resolve the numerical issue of a truncation of an infinite domain, a modified expression similar to (6) is not needed for the periodic case. The velocity $u_N(x,t)$ in terms of characteristics $q_i(t)$ and the conjugate momentum $p_i(t)$ is given by

$$u_N(x,t) = \frac{h}{2}\sum_{j=-N/2}^{N/2-1}\phi_L(x,q_j(t))p_j(t) - \kappa. \tag{15}$$

Equations (14) and (15) constitute the periodic particle algorithm. We remark that a similar algorithm but based on a different perspective has been proposed in [8].

4 The integral formulation

By using the auxiliary dependent variable

$$m(x,t) \equiv (1-\partial_x^2)\,u(x,t)\,, \tag{16}$$

shallow-water wave equation (1) can be written as

$$m_t = -2(m+\kappa)\,u_x - u m_x. \tag{17}$$

In a periodic domain, the variable u is represented in terms of m by the convolution over one period L with the periodic Green's function for the operator $(1-\partial_x^2)$. It is easy to see that such Green's function is precisely the periodic kernel ϕ_L defined by (10) and so

$$u(x,t) = \int_{-L/2}^{L/2}\phi_L(x,y)m(y,t)\,dy\,, \tag{18}$$

With (18), equation (17) can be written as an integro-differential equation:

$$\begin{aligned}m_t =& -2(m+\kappa)\,\partial_x\left(\int_{-L/2}^{L/2}\phi_L(x,y)m(y,t)\,dy\right)\\ &- m_x\int_{-L/2}^{L/2}\phi_L(x,y)m(y,t)\,dy.\end{aligned} \tag{19}$$

Alternatively, moving all the terms independent of the t-derivative in (1) to the right-hand side of this equation, yields

$$(1-\partial_x^2)u_t = -2\kappa u_x - 3uu_x + 2u_x u_{xx} + uu_{xxx}\,, \tag{20}$$

which, by the convolution formula (18), expresses u_t as

$$u_t = \int_{-L/2}^{L/2}\phi_L(x,y)\,(-2\kappa u_y - 3uu_y + 2u_y u_{yy} + uu_{yyy})\,dy. \tag{21}$$

Integration by parts along with periodic boundary conditions, then produces the integro-differential equation

$$u_t = -uu_x + \int_{-L/2}^{L/2}\psi_L(x,y)\left(u^2 + \frac{u_y^2}{2} + 2\kappa u\right)dy, \tag{22}$$

which we will refer to as the *integral formulation*.

5 Numerical examples

Example 1: The shallow water equation (1) admits periodic travelling wave solutions $u(x,t) = U(x-ct)$ expressed implicitly by the elliptic function $\Pi(\varphi,\alpha^2,k)$ [3]. A specific example can be provided by taking the mean mass of the wave to be such that the minima of u are located at $u=0$, and the wave elevation is positive. In this case one finds the solution of the travelling wave equation

$$U' = \pm\sqrt{\frac{-U^3 + (c-2\kappa)U^2 + C(A)\,U}{c-U}} \tag{23}$$

where the integration constant $C(A)$ is a function of the wave amplitude A, in the form

$$x = \frac{2}{\sqrt{a_1(a_2-a_3)}}(a_1-a_2)\Pi(\varphi,\alpha^2,k)\,. \tag{24}$$

Here φ is a function of the dependent variable U, while the constants a_i, $i=1,2,3$ (ordered in magnitude as $a_3 < 0 < a_2 < a_1$), and the parameters k and α are defined in terms of the wave velocity c, and the amplitude A, respectively. The relations are:

$$\varphi = \arcsin\left(\sqrt{\frac{a_1}{a_2}\,\frac{a_2-U}{a_1-U}}\right),$$

$$\begin{aligned}A \equiv a_3 =& \frac{1}{2}\left(c-2\kappa-\sqrt{(c-2\kappa)^2+4C}\right)\\ a_2 =& \frac{1}{2}\left(c-2\kappa+\sqrt{(c-2\kappa)^2+4C}\right)\\ a_1 =& c\,,\end{aligned}$$

and

$$k = \sqrt{\frac{a_2}{a_1}\,\frac{a_1-a_3}{a_2-a_3}}, \quad \alpha = \sqrt{\frac{a_2}{a_1}}\,.$$

Thus, the wavelength L of this periodic solution is linked to the parameter κ, the wave speed c, and the wave amplitude A by the relation

$$L = \frac{4}{\sqrt{a_1(a_2-a_3)}}(a_1-a_2)\Pi(\phi,\alpha^2,k)\,.$$

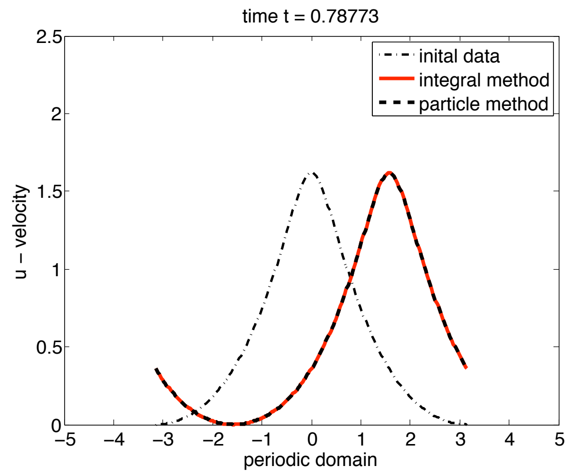

Figure 1: A snap shot of the traveling solution.

While solution (24) may not have a direct physical interpretation, it can be used to test numerical schemes in the dispersive case, as illustrated next.

If we consider the following parameters: $c = 2$, $\kappa = 1/2$, and the integration constant $C = 1$, which determine the wavelength (period) $L \approx 6.3019$. The initial data (waveform) of the traveling wave solution, with data points $N = 256$, is the dot-dash line shown in Figure 1. Both the particle method and the integral formulation described in Section 3 and 4, respectively, are used to evolve the traveling wave initial condition in time. Figure 1 also shows the snapshot of the traveling wave around time $t = 0.788$. The solid line is the solution computed by the integral formulation, while the dash line is the one computed by the particle method. They are visually indistinguishable.

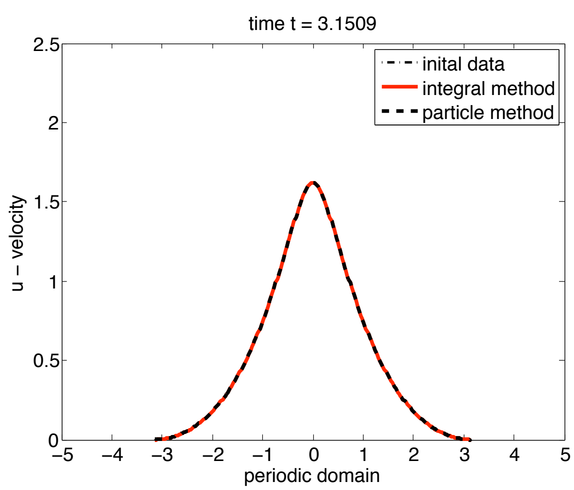

Figure 2: Traveling wave over one period.

Figure 2 demonstrates that the wave travels at the correct speed over one period. With the length of the periodic domain $L \approx 6.3019$, it takes time $t = 3.1509$ for the wave to travel through the domain. The wave moves at the speed $c = 2$, as expected. In addition, Figure 2 plots the final waveforms on top of the initial one. It shows that both schemes preserve the traveling waveform over this evolution time. The time step used in this calculation is $\Delta t = 0.0158$, while the grid size is $h = 0.0246$.

Table 1 shows the grid refinement study for the particle method, and Table 2 is the counterpart for the integral formulation. We compute the error between each two grids with the (finite) l_2-norm

$$\|\epsilon\| = \sqrt{\frac{1}{N}\sum_{j=1}^{N} \epsilon_j^2}. \tag{25}$$

Since the error goes down by four (in average) when we refine the grid, it provides evidence that both methods are second-order accurate.

N	64	128	256	512	1024
$\|\|u_N - u_{N/2}\|\|$		1.15e-3	3.37e-4	9.31e-5	2.21e-5
ratio			3.42	3.62	4.21

Table 1: Convergence rate for the particle method.

N	64	128	256	512	1024
$\|\|u_N - u_{N/2}\|\|$		1.17e-3	6.75e-4	1.68e-4	4.22e-5
ratio			2.55	4.00	3.99

Table 2: Convergence rate for the integral formulation.

Example 2: The second example is the initial value problem of the non-dispersive case, $\kappa = 0$, in a periodic domain. For this case it may be expected that the solution evolves free of shocks for all times (a priori bounds on the initial condition [4,5] which ensure that a vertical slope is achieved in finite time at inflection points are in fact violated for the initial conditions we consider). However, for the dispersionless case $\kappa = 0$ the numerical simulations in [4,6,7] show that a rather sharply peaked solitary wave forms and moves away from the origin. From the viewpoint of the particle method, the peaked solution arises from particles clustering rapidly in the region of the peak of the solitary wave(s) [4,6,7]. We provide an example of a simulation in the $\kappa = 0$ case using the initial condition $u_0(x) = \mathrm{sech}(x)$. Embedding this non-periodic initial condition in the large periodic domain $L = 40$ with $N = 800$ particles yields Figure 3. This shows that the primary peaked solution traveling to the right has propagated into the periodic domain from the left. The solid line is computed by the integral formulation and the dash line is from the particle method. Discrepancy can be observed at the secondary emerging peaked solution between the two methods. To investigate this discrepancy, Figure 4 presents a grid refinement study for the particle method. Results from three different grids at time $t = 8$ are reported. The computed solutions are consistent between grids for the particle method, whereas the analogous convergence study shown in Figure 5 for the integral formulation clearly indicates lack of convergence on these grid sizes. Thus, the figure shows that the numerical approximation using the integral method appears to converge to the solid line, which is the result computed by the finest-grid particle method. This finding of slow convergence of the integral formulation for the dispersionless case is consistent with that reported in [6,7] for the infinite line boundary conditions. We remark that the choice of parameters for the initial condition in this example is such that the period is much larger than the scale of the region where the initial pulse is concentrated, so that the initial dynamics illustrated by Figure 4 and 5 resembles closely the one observed for its infinite line counterpart [6,7].

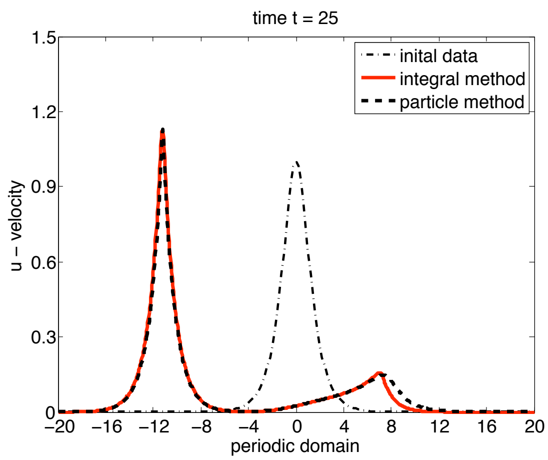

Figure 3: Study for $\kappa = 0$, the non-dispersive case.

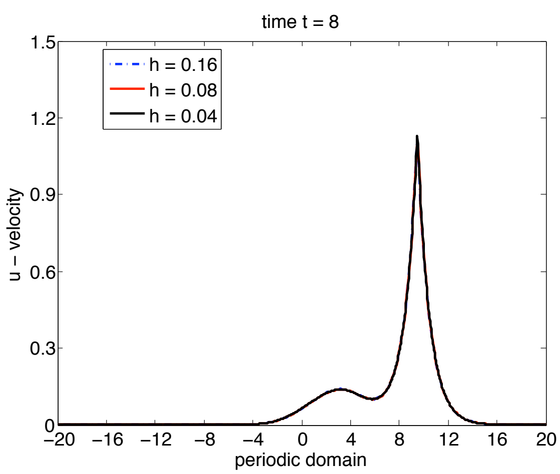

Figure 4: Grid refinement study for the particle method.

We note that to evaluate the integrals in both particle and integral method require $O(N^2)$ operation counts, where N is the number of particles. A fast summation algorithm analogous to that developed in [6,7] can be introduced to reduce the computational cost of evaluating the integrals from $O(N^2)$ to $O(N)$. We expect to report on this fast summation algorithm in a subsequent paper.

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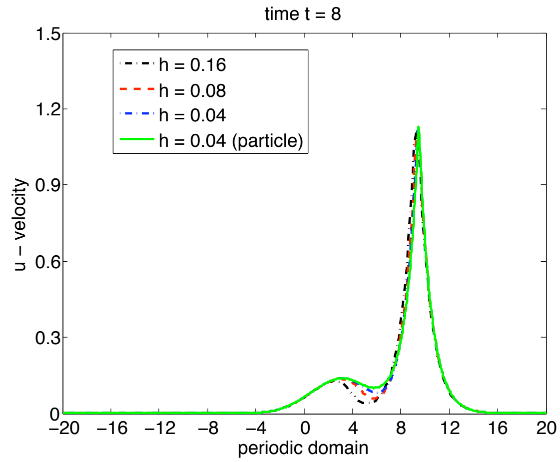

Figure 5: Grid refinement study for the integral method.

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