DIMENSIONAL ANALYSIS
(Physics 1210 Notes Appendix E)

1. PREFACE

All terms that are separated by equal, plus and minus signs in any valid relationship must obviously have the same units. On a more fundamental basis, all terms in an equation must also contain the same dimensions. This may be a trivial observation but it is a powerful tool in checking many of our day to day calculations. Dimensional considerations also have important implications in both experimental testing and theoretical analysis. The primary dimensions are mass (*m*), length (l), time (*t*), temperature (*T*), charge (*e*) and mole (*mol*). Dimensions are independent of the units that are used to measure them. For instance, some commonly used units for length (l) are, μm, m, km and light year.

2. DIMENSIONAL ANALYSIS

As an example of dimensional analysis (which is commonly called the Buckingham π theorem), we will utilize the fluid dynamics experiment that is described in the Physics 1210 manual (Experiment 14). In this experiment a sphere was dropped through a viscous fluid and Stokes' law was utilized to determine the fluid's viscosity. In this illustration assume that the drag relationship for a sphere is unknown, including the theoretically derived Stokes' drag law, which has a very limited region of application. The first task is to assemble the pertinent parameters that the drag should depend upon. Let's speculate that

$$F_{drag} = F_{drag}\left(D, V, \rho, \mu, T\right) \quad (1)$$

where

F_{drag} = steady state drag on a sphere moving through a fluid (ml/t^2)
D = sphere diameter (l)
V = velocity of the sphere relative to the fluid (l/t)
= fluid density (m/l^3)
= dynamic viscosity of fluid (m/lt)
T = fluid temperature (T)

Rewrite the above equation in the form

$$\Phi\left(F_{drag}, D, V, \rho, \mu, T\right) = 0 \quad (2)$$

which implies that the above variables must combine in a manner to produce a dimensionless relationship since the right side of the equation is dimensionless. In other words, there must exist an interdependence between the set of power coefficients (*a,b,c,d,e,f)* such that the parameter

$$\left(\left(F_{drag}\right)^a, D^b, V^c, \rho^d, \mu^e, T^f\right) \quad (3)$$

is dimensionless. Replacing the variables with their respective dimensions produces the result

$$\left(\frac{ml}{t^2}\right)^a l^b \left(\frac{l}{t}\right)^c \left(\frac{m}{l^3}\right)^d \left(\frac{m}{lt}\right)^e T^f \quad (4)$$

which implies

m:	a			$+ d$	$+ e$		$= 0$
l:	a	$+ b$	$+ c$	$-3d$	$- e$		$= 0$
t	$-2a$		$- c$			$- e$	$= 0$
T						f	$= 0$

The above equations indicate that temperature cannot be a pertinent variable since ($f = 0$) in this analysis. This variable should not have been included in the first place since viscosity is a function of the fluid's temperature and density, and this dependence means that only two of these variables can be chosen to be independent. This leaves 3 (j) equations in 5 (i) unknowns. The number of dimensionless groups that can be formed is

$$\frac{i!}{(j+1)!(i-j-1)!} \tag{5}$$

i.e. 5 in this example but all of these are not independent. The Buckingham π theorem states that there is at least **one set of only ($i - j$) independent dimensionless variables,** 2 in this case.

To close the above set of equations, two constants must be assigned values. Let $a = 1$ to produce a dimensionless group that involves the dependent variable F_{drag} and arbitrarily set "e" to 0. Solving the resulting set of equations produces the first dimensionless group that is usually labeled π_1:

$$\pi_1 = \frac{F_{drag}}{\rho V^2 D^2} \tag{6}$$

Multiplying this parameter by $8/\pi$ converts it into the drag coefficient C_D that is commonly used in fluid dynamics. (Why $8/\pi$?) A second group can be formed by reversing the values given to "a" and "e" ($a = 0$ & $e = 1$). This produces the result

$$\pi_2 = \frac{\mu}{\rho V D} \tag{7}$$

which is used in its inverted form and is called the Reynolds number Re_D.

The final result is $\Phi(C_D, Re_D) = 0$ or $C_D = \Psi(Re_D)$. Thus we have reduced an unknown functional relationship between five variables to an unknown relationship between two dimensionless groups, which can be experimentally investigated and, in some rare cases, analytically determined. This result is very significant because it implies that two experiments should produce the same drag coefficient as long as they have the same Reynolds number -- even if the fluids, sphere diameter and velocities are quite different. This forms the whole basis of model testing; for instance, the use of small scale models in wind tunnels to simulate large aircraft.

In forming the list of independent variables, suppose a superfluous variable is included. This would be detected when the dependent variable does not vary when the value of the superfluous variable is changed while the other independent variables are held constant. What happens when a variable is not included in the independent variable list? In the above example, suppose we had left out density ($d = 0$). Then the Buckingham theorem implies that there is only one independent dimensionless group in this case which is determined to be $\pi_1 = F_{drag}/(DV\mu)$ by setting $a = 1$ in the m, l & t equations. This indicates that this parameter must be constant since $\Phi(\pi_1) = 0$ which would not be substantiated by experiment.

3. REFERENCES

1. Eide, A.R., R.D. Jenison, L.H. Mashaw & L.L. Northup, *Engineering Fundamentals and Problem Solving*, McGraw-Hill, 1989, p 119-137.

2. Becker, H.A., *Dimensionless parameters: Theory and Methodology*, John Wiley & Sons, 1976.

3. Munson, B.R., D.F. Young & T.H. Okiishi, *Fundamentals of Fluid Mechanics*, John Wiley & Sons, 1990, p 404-418 & 813-815.