

# A Methodology for Numerical Solution of Ordinary Differential Equations

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An important element in the examination of many physical systems is an ability to formulate (and solve) a variety of mathematical models. Differential equations arise in many instances and the need to be able to solve these equations is important. Often, the equations do not have exact analytical solutions, so recourse to numerical solutions are necessary. As an introduction, we will discuss the numerical solution of a first order, ordinary differential equation:

$$\frac{dx}{dt} = f(x, t)$$

with the initial condition:

$$x(0) = x_0.$$

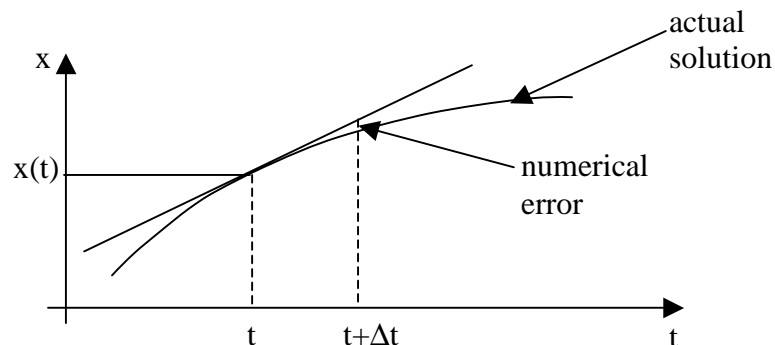
There are a number of increasingly sophisticated, stable and accurate ways to numerically "integrate" this equation forward in "time",  $t$  ( $t$  does not have to correspond to time, it is simply the independent variable). The simplest approach is an "Euler Integration", where we approximate the derivative by the finite difference approximation:

$$\frac{dx}{dt} = \frac{x(t + \Delta t) - x(t)}{\Delta t},$$

and solve for  $x(t + \Delta t)$ :

$$x(t + \Delta t) = f[x(t), t]\Delta t + x(t),$$

which is the first term of a Taylor series expansion about  $t$ . Note that in this approximation,  $f(x, t)$  is evaluated at the old time step. To see what this approximation implies, refer to Figure 1.



**Figure 1** - An Euler time-step example.

The problem with this method is apparent from this figure: there is unavoidable error (unless the actual solution is a straight line!). One "fix" is to make  $\Delta t$  small, but not so small as to be even close to the resolution of the computer. (Another possible fix would be to evaluate  $f(x,t)$  at the middle of the  $t$ -interval, but one doesn't know  $x$  at this time without further complications).

More accurate, stable methods have been developed and are preferred over this simple Euler approach (the most common workhorse is the Runge-Kutta technique, where many versions and refinements are available). Such methods are more complicated to program, so we will discuss only the Euler approach.

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An example:

Suppose:

$$\frac{dx}{dt} = f(x,t) = \frac{(t+1)}{x^2},$$

where the initial condition is specified as:  $x_0 = 1$ . The analytical solution is:

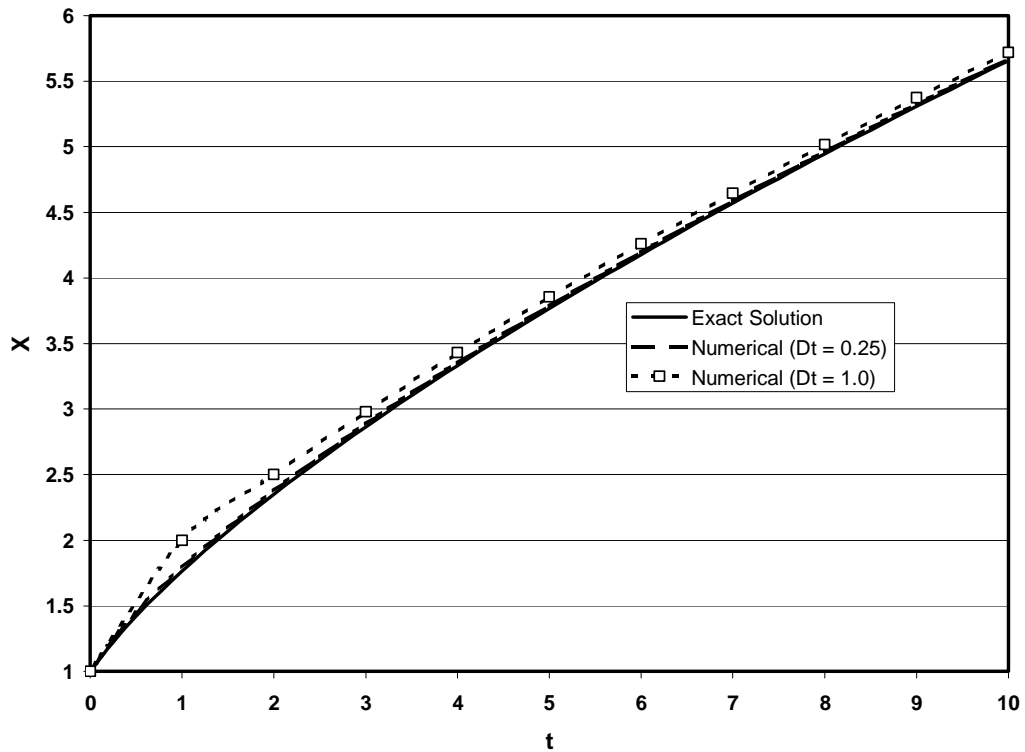
$$x = \left[ 3 \left( \frac{t^2}{2} + t + \frac{1}{3} \right) \right]^{1/3},$$

which we can use to compare to a numerical solution.

We begin by recognizing  $f(x,t) = (t+1)/x^2$ , so:

$$x(t + \Delta t) = x(t) + \left[ \frac{(t+1)}{x^2} \right] \Delta t.$$

An Excel spread-sheet solution is attached as Appendix A. The numerical solution (for various values of  $\Delta t$ ) is shown in Figure 2. We note the improved solution for the reduced step size. One of the better ways of examining the appropriateness of the chosen step size is to further reduce the step size and see if the solutions are unchanged.



**Figure 2** - Comparison of numerical and analytical solutions for the example differential equation. The numerical solutions are for increments in  $t$  of 0.25 and 1.00. For  $\Delta t = 1.0$ , note the irrecoverable initial error due to the large step-size.

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The approach for solving higher order differential equations is to break the equation into a series of coupled first-order differential equations and then to use one of the methods for solving first order equations on the resulting sets of equations. For example, a second-order differential equation may be written as:

$$\frac{d^2x}{dt^2} = f\left(\frac{dx}{dt}, x, t\right).$$

We then define  $x_1$  and  $x_2$  as:

$$x = x_1,$$

$$\frac{dx}{dt} = \frac{dx_1}{dt} = x_2,$$

so that the original differential equation may be written as:

$$\frac{dx_1}{dt} = x_2,$$

$$\frac{dx_2}{dt} = f(x_2, x_1, t).$$

These two first-order differential equations may then be solved simultaneously. Higher order equations may be approached in the same way. Additionally, coupled differential equations may be solved as additional first-order differential equations, using the same techniques:

$$\frac{d^2x}{dt^2} = f_x\left(\frac{dx}{dt}, \frac{dy}{dt}, x, y, t\right),$$

$$\frac{d^2y}{dt^2} = f_y\left(\frac{dx}{dt}, \frac{dy}{dt}, x, y, t\right).$$

In this case, we may write:

$$\frac{dx_1}{dt} = x_2,$$

$$\frac{dx_2}{dt} = f_x(x_2, y_2, x_1, y_1, t),$$

$$\frac{dy_1}{dt} = y_2,$$

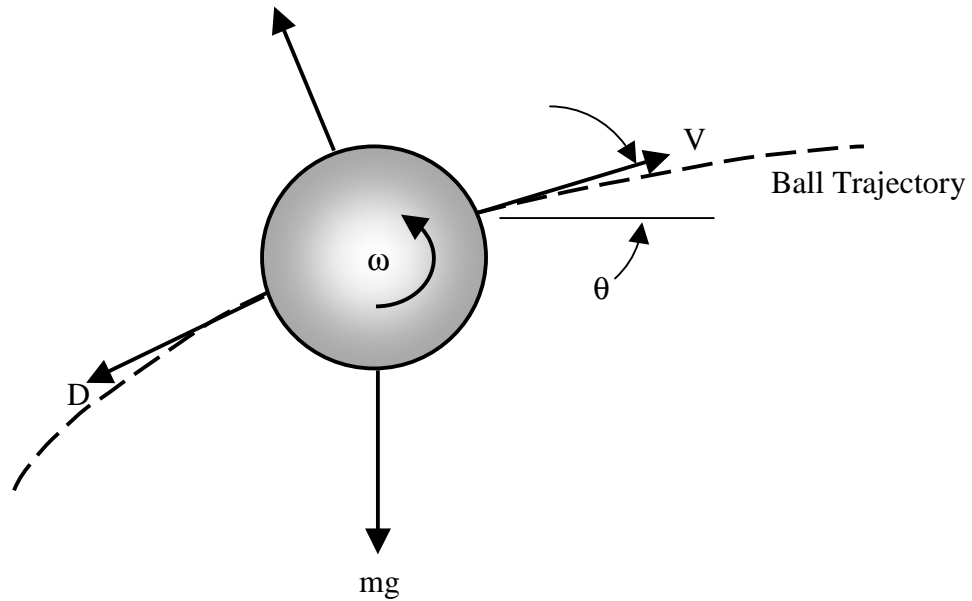
$$\frac{dy_2}{dt} = f_y(x_2, y_2, x_1, y_1, t).$$

We shall discuss this case further in a rather detailed example: the analysis of the trajectory of a spinning sphere.

**Example:** Trajectory analysis of a spinning sphere.

The geometry (and forces) associated with a sphere of mass  $m$  traveling along a trajectory under the action of gravity and aerodynamic forces is shown in Figure 3. For such a situation, Newton's Second Law of Motion may be written as:

$$m \frac{d^2 \vec{x}}{dt^2} = \Sigma \vec{F}.$$



**Figure 3** – Identification of forces acting on a translating, spinning sphere, under the influence of aerodynamic forces and gravity.

This vector equation may be broken down into its vector components as:

$$m \frac{d^2 x}{dt^2} = -D \cos \theta - L \sin \theta,$$

$$m \frac{d^2 y}{dt^2} = -D \sin \theta + L \cos \theta - mg,$$

where:

$$\theta = \tan^{-1}\left(\frac{V_y}{V_x}\right) = \tan^{-1}\left(\frac{dy/dt}{dx/dt}\right),$$

$$D = C_D A_p \left(\frac{1}{2} \rho V^2\right),$$

$$L = C_L \rho \omega V R^3,$$

$$\omega = \omega_0 e^{-t/t_0},$$

$$V^2 = V_x^2 + V_y^2.$$

There is a lot of terminology in these equations that needs to be mentioned, for example:

$L, D$  = aerodynamic lift and drag forces, respectively, on the sphere,

$C_D$  = drag coefficient,

$C_L$  = lift coefficient (the lift will be positive in the figure if the spin has the sense of rotation as shown),

$A_p$  = projected area (normal to the velocity vector) [ $A_p = \pi R^2$ ],

$\rho$  = fluid density,

and the spin of the ball is modeled by an exponentially decaying spin behavior, where:

$$\omega = \omega_0 e^{-t/t_0},$$

$\omega_0$  = initial spin rate,

$t_0$  = spin decay time.

The series of differential equations that need to be solved in this case are:

$$\frac{dx_1}{dt} = x_2,$$

$$\frac{dx_2}{dt} = -\frac{1}{m} [D \cos \theta + L \sin \theta],$$

$$\frac{dy_1}{dt} = y_2,$$

$$\frac{dy_2}{dt} = -\frac{1}{m} [D \sin \theta - L \cos \theta + mg],$$

where the lift and drag terms are now written as:

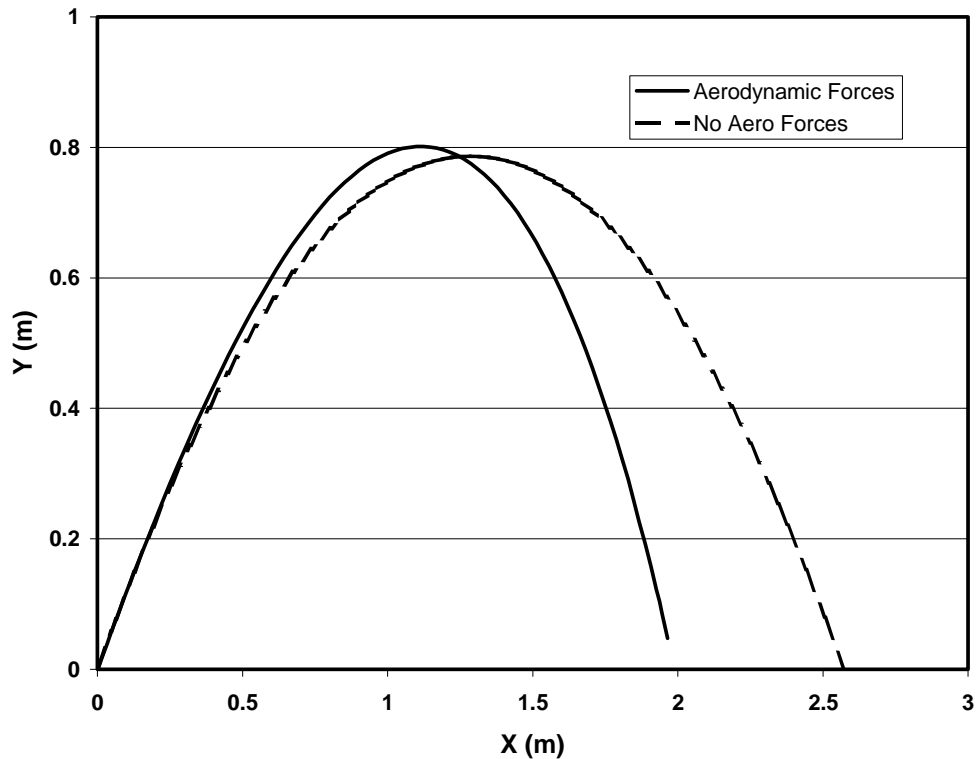
$$L = C_L \rho \omega(t) V R^3,$$

$$D = C_D A_p \left( \frac{1}{2} \rho V^2 \right),$$

$$V = (x_2^2 + y_2^2)^{1/2}.$$

Example Excel sheets that execute this numerical integration are included in Appendix B. Figure 4 illustrates some example trajectories that are obtained from this approach. There are some unrealistic parameters that have been included in these calculations, so the reader is cautioned to not take the quantitative results very seriously.

The problem with using Excel as a “forward integration” program is the need to use a variable number of cells, depending on the step size and the trajectory results (i.e. terminating when  $y < 0$ ). In this case, the program requires a continuing interaction by the user to copy only the appropriate cells into a plot, etc. There are certainly better computer tools for the solution of ordinary differential equations, but the current methodology illustrates the general approach and is pretty easy to implement.



**Figure 4** – Example trajectories for a translating, spinning sphere (with and without aerodynamic forces). The specific properties for these cases are documented in Appendix B.

## **Appendices**

Appendix A – Excel sheets showing the implementation of the Euler integration technique for the example first order differential equation.

Appendix B – Excel sheets showing the implementation of the Euler integration technique for the example spinning sphere trajectory analysis.



**Appendix A –Euler integration technique for first order differential equations.**

**Example Euler Integration**

$$\frac{dx}{dt} = \frac{(t+1)}{x^2}$$

Dt= 1

t	x-analytical	x-numerical
0	=(A6*A6/2+A6+1)	1
=A6+Dt	=(3*(A7*A7/2+A7+1/3))^(1/3)	=C6+funky(C6,A6)*Dt
=A7+Dt	=(3*(A8*A8/2+A8+1/3))^(1/3)	=C7+funky(C7,A7)*Dt
=A8+Dt	=(3*(A9*A9/2+A9+1/3))^(1/3)	=C8+funky(C8,A8)*Dt
=A9+Dt	=(3*(A10*A10/2+A10+1/3))^(1/3)	=C9+funky(C9,A9)*Dt
=A10+Dt	=(3*(A11*A11/2+A11+1/3))^(1/3)	=C10+funky(C10,A10)*Dt
=A11+Dt	=(3*(A12*A12/2+A12+1/3))^(1/3)	=C11+funky(C11,A11)*Dt
=A12+Dt	=(3*(A13*A13/2+A13+1/3))^(1/3)	=C12+funky(C12,A12)*Dt
=A13+Dt	=(3*(A14*A14/2+A14+1/3))^(1/3)	=C13+funky(C13,A13)*Dt
=A14+Dt	=(3*(A15*A15/2+A15+1/3))^(1/3)	=C14+funky(C14,A14)*Dt
=A15+Dt	=(3*(A16*A16/2+A16+1/3))^(1/3)	=C15+funky(C15,A15)*Dt
=A16+Dt	=(3*(A17*A17/2+A17+1/3))^(1/3)	=C16+funky(C16,A16)*Dt
=A17+Dt	=(3*(A18*A18/2+A18+1/3))^(1/3)	=C17+funky(C17,A17)*Dt
=A18+Dt	=(3*(A19*A19/2+A19+1/3))^(1/3)	=C18+funky(C18,A18)*Dt
=A19+Dt	=(3*(A20*A20/2+A20+1/3))^(1/3)	=C19+funky(C19,A19)*Dt
=A20+Dt	=(3*(A21*A21/2+A21+1/3))^(1/3)	=C20+funky(C20,A20)*Dt
=A21+Dt	=(3*(A22*A22/2+A22+1/3))^(1/3)	=C21+funky(C21,A21)*Dt
=A22+Dt	=(3*(A23*A23/2+A23+1/3))^(1/3)	=C22+funky(C22,A22)*Dt
=A23+Dt	=(3*(A24*A24/2+A24+1/3))^(1/3)	=C23+funky(C23,A23)*Dt
=A24+Dt	=(3*(A25*A25/2+A25+1/3))^(1/3)	=C24+funky(C24,A24)*Dt
=A25+Dt	=(3*(A26*A26/2+A26+1/3))^(1/3)	=C25+funky(C25,A25)*Dt
=A26+Dt	=(3*(A27*A27/2+A27+1/3))^(1/3)	=C26+funky(C26,A26)*Dt
=A27+Dt	=(3*(A28*A28/2+A28+1/3))^(1/3)	=C27+funky(C27,A27)*Dt
=A28+Dt	=(3*(A29*A29/2+A29+1/3))^(1/3)	=C28+funky(C28,A28)*Dt
=A29+Dt	=(3*(A30*A30/2+A30+1/3))^(1/3)	=C29+funky(C29,A29)*Dt
=A30+Dt	=(3*(A31*A31/2+A31+1/3))^(1/3)	=C30+funky(C30,A30)*Dt
=A31+Dt	=(3*(A32*A32/2+A32+1/3))^(1/3)	=C31+funky(C31,A31)*Dt

**Attachment A.1.** Excel formula sheet for the example first order differential equation.

### Example Euler Integration

$$\frac{dx}{dt} = \frac{(t+1)}{x^2}$$

$$Dt = 1$$

t	x-analytical	x-numerical
0	1	1
1	1.7652	2.0000
2	2.3513	2.5000
3	2.8643	2.9800
4	3.3322	3.4304
5	3.7681	3.8553
6	4.1793	4.2590
7	4.5709	4.6449
8	4.9461	5.0157
9	5.3074	5.3734
10	5.6567	5.7198
11	5.9954	6.0560
12	6.3247	6.3832
13	6.6456	6.7023
14	6.9589	7.0139
15	7.2653	7.3188
16	7.5654	7.6175
17	7.8595	7.9105
18	8.1483	8.1982
19	8.4320	8.4808
20	8.7110	8.7589
21	8.9856	9.0326
22	9.2560	9.3023
23	9.5226	9.5681
24	9.7854	9.8302
25	10.0448	10.0890
26	10.3009	10.3444
27	10.5538	10.5967
28	10.8037	10.8461
29	11.0507	11.0926
30	11.2950	11.3364

**Attachment A.2** - Excel sheet for example first order differential equation.

```
Private Function funky(x, t)
    funky = ((t + 1) / x ^ 2)
End Function
```

**Attachment A.3** - Excel Visual Basic macro for the example first order differential equation.

**Appendix B –Euler integration technique for the example spinning sphere trajectory analysis.**

Initial Angle	50 degrees
Initial Ball Rotation	50 rad/s
Ball spin-down rate	1 s <sup>-1</sup>
Air Density	0 kg/m <sup>3</sup>
Time Increment	0.02 s
Ball mass	0.005 kg
Ball radius	0.022 m

<b>time (sec)</b>	<b>x (m)</b>	<b>y (m)</b>	<b>Vx (m/s)</b>	<b>Vy (m/s)</b>
0.00	0.000	0.000	3.21	3.83
0.02	0.064	0.077	3.21	3.63
0.04	0.129	0.149	3.21	3.44
0.06	0.193	0.218	3.21	3.24
0.08	0.257	0.283	3.21	3.05
0.10	0.321	0.344	3.21	2.85
0.12	0.386	0.401	3.21	2.65
0.14	0.450	0.454	3.21	2.46
0.16	0.514	0.503	3.21	2.26
0.18	0.579	0.548	3.21	2.06
0.20	0.643	0.589	3.21	1.87
0.22	0.707	0.627	3.21	1.67
0.24	0.771	0.660	3.21	1.48
0.26	0.836	0.690	3.21	1.28
0.28	0.900	0.715	3.21	1.08
0.30	0.964	0.737	3.21	0.89
0.32	1.028	0.755	3.21	0.69
0.34	1.093	0.769	3.21	0.49
0.36	1.157	0.779	3.21	0.30
0.38	1.221	0.784	3.21	0.10
0.40	1.286	0.787	3.21	-0.09
0.42	1.350	0.785	3.21	-0.29
0.44	1.414	0.779	3.21	-0.49
0.46	1.478	0.769	3.21	-0.68
0.48	1.543	0.755	3.21	-0.88
0.50	1.607	0.738	3.21	-1.07
0.52	1.671	0.716	3.21	-1.27
0.54	1.736	0.691	3.21	-1.47
0.56	1.800	0.662	3.21	-1.66
0.58	1.864	0.628	3.21	-1.86
0.60	1.928	0.591	3.21	-2.06
0.62	1.993	0.550	3.21	-2.25
0.64	2.057	0.505	3.21	-2.45
0.66	2.121	0.456	3.21	-2.64
0.68	2.185	0.403	3.21	-2.84
0.70	2.250	0.346	3.21	-3.04
0.72	2.314	0.286	3.21	-3.23
0.74	2.378	0.221	3.21	-3.43
0.76	2.443	0.152	3.21	-3.63
0.78	2.507	0.080	3.21	-3.82
0.80	2.571	0.003	3.21	-4.02
0.82	2.635	-0.077	3.21	-4.21
<b>0.84</b>	<b>2.700</b>	<b>-0.161</b>	<b>3.21</b>	<b>-4.41</b>

**Attachment B.1** - Excel sheet for example second-order differential equation (trajectory analysis problem).

**A Dynamical Model of a Sphere's Trajectory**

**Input parameters (somewhat appropriate for wiffle balls)**

Initial Velocity	5	m/s
Initial Angle	50	degrees
Initial Ball Rotation	50	rad/s
Ball spin-down rate	1	s <sup>-1</sup>
Air Density	0	kg/m <sup>3</sup>
Time Increment	0.02	s
Ball mass	0.005	kg
Ball radius	0.022	m

time	x	y	Vx	Vy
0	0	0	=C\$37*COS(C\$38*PI()/180)	=C\$37*SIN(C\$38*PI()/180)
=A47+dt	=D47*dt+B47	=E47*dt+C47	=D47+funcx(A47,B47,D47,C47,E47,mass,rho,radius,omega_o,to)*dt	=E47+funcy(B47,C47,E47,D47,F47,mass,rho,radius,omega_o,to)*dt
=A48+dt	=D48*dt+B48	=E48*dt+C48	=D48+funcx(A48,B48,D48,C48,E48,mass,rho,radius,omega_o,to)*dt	=E48+funcy(B48,C48,E48,D48,F48,mass,rho,radius,omega_o,to)*dt
=A49+dt	=D49*dt+B49	=E49*dt+C49	=D49+funcx(A49,B49,D49,C49,E49,mass,rho,radius,omega_o,to)*dt	=E49+funcy(B49,C49,E49,D49,F49,mass,rho,radius,omega_o,to)*dt
=A50+dt	=D50*dt+B50	=E50*dt+C50	=D50+funcx(A50,B50,D50,C50,E50,mass,rho,radius,omega_o,to)*dt	=E50+funcy(B50,C50,E50,D50,F50,mass,rho,radius,omega_o,to)*dt
=A51+dt	=D51*dt+B51	=E51*dt+C51	=D51+funcx(A51,B51,D51,C51,E51,mass,rho,radius,omega_o,to)*dt	=E51+funcy(B51,C51,E51,D51,F51,mass,rho,radius,omega_o,to)*dt
=A52+dt	=D52*dt+B52	=E52*dt+C52	=D52+funcx(A52,B52,D52,C52,E52,mass,rho,radius,omega_o,to)*dt	=E52+funcy(B52,C52,E52,D52,F52,mass,rho,radius,omega_o,to)*dt
=A53+dt	=D53*dt+B53	=E53*dt+C53	=D53+funcx(A53,B53,D53,C53,E53,mass,rho,radius,omega_o,to)*dt	=E53+funcy(B53,C53,E53,D53,F53,mass,rho,radius,omega_o,to)*dt
=A54+dt	=D54*dt+B54	=E54*dt+C54	=D54+funcx(A54,B54,D54,C54,E54,mass,rho,radius,omega_o,to)*dt	=E54+funcy(B54,C54,E54,D54,F54,mass,rho,radius,omega_o,to)*dt
=A55+dt	=D55*dt+B55	=E55*dt+C55	=D55+funcx(A55,B55,D55,C55,E55,mass,rho,radius,omega_o,to)*dt	=E55+funcy(B55,C55,E55,D55,F55,mass,rho,radius,omega_o,to)*dt
=A56+dt	=D56*dt+B56	=E56*dt+C56	=D56+funcx(A56,B56,D56,C56,E56,mass,rho,radius,omega_o,to)*dt	=E56+funcy(B56,C56,E56,D56,F56,mass,rho,radius,omega_o,to)*dt
=A57+dt	=D57*dt+B57	=E57*dt+C57	=D57+funcx(A57,B57,D57,C57,E57,mass,rho,radius,omega_o,to)*dt	=E57+funcy(B57,C57,E57,D57,F57,mass,rho,radius,omega_o,to)*dt
=A58+dt	=D58*dt+B58	=E58*dt+C58	=D58+funcx(A58,B58,D58,C58,E58,mass,rho,radius,omega_o,to)*dt	=E58+funcy(B58,C58,E58,D58,F58,mass,rho,radius,omega_o,to)*dt

**Attachment B.2** - Excel formula sheet for example second-order differential equation (trajectory analysis).

```

Private Function funcx(t, x1, x2, y1, y2, m, rho, r, omega_bo, t_o)
    Cd = 1.5
    Cl = 0.5
    Pi = 3.141592654
    V = Sqr(x2 ^ 2 + y2 ^ 2) 'velocity magnitude
    omega_b = omega_bo * Exp(t / t_o) 'ball spin rate
    D = 0.5 * rho * V ^ 2 * Cd * Pi * r ^ 2 'Drag
    L = 0.5 * rho * omega_bo * Cl * V * r ^ 3 'Lift
    theta = Atn(y2 / x2) 'tangent trajectory angle
    funcx = -(D * Cos(theta) + L * Sin(theta)) / m
End Function

```

```

Private Function funcy(t, x1, x2, y1, y2, m, rho, r, omega_bo, t_o)
    g = 9.81
    Cd = 1.5
    Cl = 1.5
    Pi = 3.141592654
    V = Sqr(x2 ^ 2 + y2 ^ 2) 'velocity magnitude
    omega_b = omega_bo * Exp(t / t_o) 'ball spin rate
    D = 0.5 * rho * V ^ 2 * Cd * Pi * r ^ 2 'Drag
    L = 0.5 * rho * omega_bo * Cl * V * r ^ 3 'Lift
    theta = Atn(y2 / x2) 'tangent trajectory angle
    funcy = -(D * Sin(theta) - L * Cos(theta) + m * g) / m
End Function

```

**Attachment B.3** - Excel Visual Basic Function macros for example second-order differential equation (trajectory analysis problem).