APPENDIX B
PARTIAL DIFFERENTIATION

The *ordinary* derivative of a function $f(x)$ with respect to the (only) independent variable x, is defined as

$$\frac{df}{dx} = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \tag{1}$$

When the function f has more than one independent variable, like $f(x,y)$, where x and y are both independent variables, one defines the *partial* derivative of the function with respect to x as

$$\frac{df}{dx} = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon, y) - f(x,y)}{\varepsilon} \tag{2}$$

Similarly, the partial derivative with respect to y is defined as

$$\frac{df}{dy} = \lim_{\varepsilon \to 0} \frac{f(x, y+\varepsilon) - f(x,y)}{\varepsilon} \tag{3}$$

where the funny symbol $\frac{\partial}{\partial x}$ is used instead of the ordinary $\frac{d}{dx}$ as a reminder that there are independent variables other than x, and that all, except x, are being held constant. So this is the main idea: When taking the partial derivative of a function with respect to one variable, treat all the rest as if they were ordinary numbers, like 3 or 75!

One can define higher order derivatives $\frac{\partial^2 f}{\partial x^2}$ or $\frac{\partial^4 f}{\partial y^4}$ etc. There are also the so-called "mixed" derivatives, such as $\frac{\partial^3 f}{\partial x^2 \partial y}$. This particular one means, first take the partial with respect to y, and then with respect to x, and again with respect to x. Hence:

$$\frac{\partial^3 f}{\partial x^2 \partial y} \equiv \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right] \tag{4}$$

Before going on to an example, note that, if the function f is a "smooth" function of x and y, then the order in which the partial derivatives are taken in a mixed derivative is immaterial. Thus

$$\frac{\partial^3 f}{\partial x^2 \partial y} \equiv \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right] = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right] etc. \tag{5}$$

if $f(x,y)$ is "smooth". Most functions one encounters in applied sciences are "smooth".

EXAMPLE: $f(x,y) = 3x^3y + e^{2x}$ (This is a smooth function!)

$$\frac{\partial f}{\partial x} = 9x^2y + 2e^{2x} \ , \ \frac{\partial f}{\partial y} = 3x^3$$

$$\frac{\partial^2 f}{\partial x^2} = 18xy + 4e^{2x} \ , \ \frac{\partial^2 f}{\partial y^2} = 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[9x^2y + 2e^{2x}\right] = 9x^2 \ , \ \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[3x^3\right] = 9x^2$$

hence

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

TAYLOR SERIES

In the development of propagation of errors, one made use of Taylor's theorem which says that one can expand any continuously differentiable function in terms of a Taylor series. To see how this works, one can expand a function in terms of a power series.

$$f'(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \tag{6}$$

The trouble now lies in finding the a_ns. To do this, look around $x = 0$. At $x = 0$, all the terms that contain an x go to 0 and one has

$$a_0 = f(0) \tag{7}$$

Well, that bags one of them. Now, take the derivative of both sides of Eq. 1 with respect to x

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots \tag{8}$$

Again, if choosing $x = 0$, one gets

$$a_1 = f'(0) \tag{9}$$

Taking another derivative with respect to x yields

$$f''(x) = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 \ldots \tag{10}$$

and evaluating at $x = 0$ gives

$$a_2 = \frac{1}{2} f''(0) \tag{11}$$

Continuing this process, one sees that

$$a_n = \frac{f^{(n)}(0)}{n!} \tag{12}$$

and substituting back into the power series expansion,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \tag{13}$$

All that is required is that $f(x)$ be a continuously differentiable function.

If one wishes to expand the function about some other value of x such as $x = \bar{x}$, one simply writes the function as

$$f(x)=\sum_{n=0}^{\infty} a_n(x-\bar{x})^n \tag{14}$$

and continues as before yielding

$$f(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(\bar{x})}{n!}(x-\bar{x})^n \tag{15}$$

For a multivariable function $f(x,y)$ one proceeds similarly; first assume a power series expansion, in the following form, exists:

$$f(x,y)=a_{0,0}+a_{1,0}(x-x_0)+a_{0,1}(y-y_0)+a_{2,0}(x-x_0)^2 \tag{16}$$

$$+a_{1,1}(x-x_0)(y-y_0)+a_{02}(y-y_0)^2+\ldots$$

or more compactly

$$f(x,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{n,m}(x-x_0)^n(y-y_0)^m \tag{17}$$

and then proceed to find the coefficients $a_{n,m}$.
To find $a_{n,m}$:

0,0) Set $x = x_0$ and $y = y_0$ on both sides
Then $a_{0,0}=f(x_0,y_0)$

1,0) Take the 1st partial derivative with respect to x, and then set $x = x_0$, $y = y_0$
Then $a_{1,0}=\left.\frac{\partial f}{\partial x}\right|_{x_0,y_0}$

0,1)Similarly as for 1,0 , do the same for y
Then $a_{0,1}=\left.\frac{\partial f}{\partial y}\right|_{x_0,y_0}$

By simply taking the n'th partial derivative with respect to x and the m'th partial derivative with respect to y and *then* setting $x = x_0$, $y = y_0$ on both sides, one can obtain $a_{n,m}$ for all n,m. Notice that n "counts" the power of $(x - x_0)$ while m does the same for y.

Then one gets

$$a_{n,m}=\frac{1}{n!\,m!}\left.\frac{\partial^{n+m} f}{\partial x^n \partial y^m}\right|_{x_0,y_0} \tag{18}$$