PROPAGATION OF ERRORS

1. PREFACE

It is seldom possible to directly measure the quantity *R* that an experiment is designed to determine. Instead, *R* will normally be computed from related physical quantities (*x*,*y*...) that can be experimentally measured, *R(x,y,...)*. For example, the volume of a cylinder (*V*) can be computed by determining its diameter (*D*) and length (*L*); that is *V(D, L)*. These two length measurements have errors associated with them and these errors will propagate into the computed volume. A technique for estimating how errors propagate from measured quantities into the computed results of an experiment is discussed in this treatise.

2. USING DIFFERENTIAL CALCULUS TO DETERMINE THE PROPAGATION OF ERRORS

Suppose that an experiment is performed in which only two physical quantities *(x, y)* are measured so that the desired result quantity can be computed, *R(x, y)*. As described in *Experimentation and Uncertainty Analysis for Engineers*[1], each of these measured quantities, *x* and *y*, has a best estimate and probable error associated with it. The computed experimental result, *R*, should be reported in the same manner as the measured quantities, i.e. $R = \overline{R} \pm U_{\overline{R}}$ where $U_{\overline{R}}$ is the expanded uncertainty of $\overline{R}$ at a given percent confidence, normally 95%. The goal is to develop a relationship that relates the statistics of the measured quantities, *x* and *y*, to the statistics of the result, *R*.

The best estimate of *R* is obtained by using the mean values of the independent variables, $\overline{R} \equiv R(\overline{x}, \overline{y})$. This is not equivalent to finding the mean of the calculated values, $\sum R(x_i, y_i)/n$. For example, if $R(x, y) = x + y^2$ and the following values are used:

x	*y*
1	2
1	7
4	1

$R(\overline{x}, \overline{y}) = 13.11$, while $\sum R(x_i, y_i)/n = 20.0$.

The recommendations of the ISO *Guide*[2] is that $U_{\overline{R}}$ be estimated from the total variance, i.e. the sum of the variances from systematic error and random error, times a coverage factor. The ISO *Guide*[2] recommends using the values from the *t* distribution for this coverage factor assuming that the distribution for the total errors is Gaussian. Therefore, the expanded uncertainty is:

$$U_{\overline{R}\%} = t_{\%}\sqrt{S_{\beta i}^2 + S_i^2} \qquad (1)$$

where $S_{\beta i}^2$ is the variance from systematic errors and S_i^2 is the variance from random error. Variance is a statistical value defined as:

$$S^2 = \frac{\sum_{i=1}^{n}(v_i - \overline{v})^2}{n-1} \qquad (2)$$

where v_i is an individual observation, $\bar{v}$ is the sample mean and *n* is the number of observations.

The value of an individual point, R_i, can be found by expanding $R(\bar{x}, \bar{y})$ about $\bar{x}$ and $\bar{y}$ using a Taylor series.

$$R_i = R(\bar{x}, \bar{y}) + \frac{\partial R}{\partial x}(x_i - \bar{x}) + \frac{\partial R}{\partial y}(y_i - \bar{y}) + \text{Remainder} \tag{3}$$

The remainder contains second and higher order derivatives and error terms. Assuming the derivatives are of reasonable values and the errors are small, the remainder will approach zero much faster than the first order terms, and can be neglected. This approximation assumes *R(x, y)* is a continuous and differentiable function. Squaring both sides of the equation, summing over *i* and dividing both sides of the equation by (*n*-1) gives:

$$\frac{1}{n-1}\sum_{i=1}^{n}(R_i - \bar{R})^2 \approx \left(\left.\frac{\partial R}{\partial x}\right|_{\bar{x},\bar{y}}\right)^2 \left(\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2\right) + \left(\left.\frac{\partial R}{\partial y}\right|_{\bar{x},\bar{y}}\right)^2 \left(\frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2\right)$$
$$+ \frac{2}{n-1}\left.\left(\frac{\partial R}{\partial x}\frac{\partial R}{\partial y}\right)\right|_{\bar{x},\bar{y}} \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) \tag{4}$$

The last expression in the above equation is called the covariance term. This term should approach zero if the independent variables *x* and *y* are statistically independent since there are approximately the same number of positive and negative deviations from the mean. From Equation (2) the LHS of the equation can be recognized as the variance of $\bar{R}$ and the variance of $\bar{x}$ and $\bar{y}$ can be identified in the RHS of the equation. The variance of $\bar{x}$ and $\bar{y}$ are assumed to be the total variance. Substituting these values yields:

$$S_{\bar{R}}^2 \approx \left(\left.\frac{\partial R}{\partial x}\right|_{\bar{x},\bar{y}}\right)^2 S_{\bar{x}}^2 + \left(\left.\frac{\partial R}{\partial y}\right|_{\bar{x},\bar{y}}\right)^2 S_{\bar{y}}^2 \tag{5}$$

Multiplying both sides of the equation by the square of the table *t* value for the given percent confidence gives:

$$t_{\%}^2 S_{\bar{R}}^2 \approx \left(\left.\frac{\partial R}{\partial x}\right|_{\bar{x},\bar{y}}\right)^2 t_{\%}^2 S_{\bar{x}}^2 + \left(\left.\frac{\partial R}{\partial y}\right|_{\bar{x},\bar{y}}\right)^2 t_{\%}^2 S_{\bar{y}}^2 \tag{6}$$

Finally, substituting Equation (1) and taking the square root of both sides of the equation yields an expression for the probable error of the computed R .

$$U_{\bar{R}} \approx \sqrt{\left(\left.\frac{\partial R}{\partial x}\right|_{\bar{x},\bar{y}}\right)^2 U_{\bar{x}}^2 + \left(\left.\frac{\partial R}{\partial y}\right|_{\bar{x},\bar{y}}\right)^2 U_{\bar{y}}^2} \tag{7}$$

The extension of Equation (7) to more than two independent variables should be obvious.

3. ANALYTIC EXAMPLE

Using the example of the volume of a cylinder, $\bar{V} = \pi \bar{D}^2 \bar{L} / 4$ cited above, taking the partial derivatives of $\bar{V}$ with respect to $\bar{D}$ and $\bar{L}$

$$\left.\frac{\partial \bar{V}}{\partial \bar{D}}\right|_{\bar{D},\bar{L}} = \pi \bar{D}\bar{L} / 2$$ where $\bar{L}$ is held constant in this partial differential and

$$\left.\frac{\partial \bar{V}}{\partial \bar{L}}\right|_{\bar{D},\bar{L}} = \pi \bar{D}^2 / 4$$ where $\bar{D}$ is constant in this differential.

The uncertainty $U_{\bar{V}}$ can now be written in terms of the mean values of D and L and their respective uncertainties.

$$U_{\bar{V}}^{\ 2} \approx (\pi \bar{D}\bar{L}/2\ )^2 U_{\bar{D}}^{\ 2} + (\pi \bar{D}^2/4\ )^2 U_{\bar{L}}^{\ 2} \tag{8}$$

Thus the determination of $V = \bar{V} \pm U_{\bar{V}}$ is complete.

4. SYMBOLIC LOGIC PROGRAMS

Evaluation of the partial derivatives in the above example is simple and straight forward, but this is not usually the case. Most propagation of error analyses are quite complicated with many independent measurements involved and complex mathematical relationships between the dependent and independent variables. Symbolic logic programs like *DERIVE* and *Maple* are capable of evaluating the partial derivatives that are required in these analyses and therefore help eliminate the drudgery associated with these calculations. Finding these partial derivatives is quite useful in experimental design, since the sensitivity of the probable error of the calculated results to the accuracy of the various measurements that will be made can be investigated ahead of time.

5. NUMERICAL APPROXIMATION

Since any particular measurement, x_i, is the sum of the mean value plus the total associated error, $x_i = \bar{x} + U_{\bar{x}}$, an approximation of the contribution of $U_{\bar{x}}$ to the overall $U_{\bar{R}}$ can be found by adding $U_{\bar{x}}$

to $\bar{x}$, computing the resulting R using $(\bar{x}+U_{\bar{x}})$, subtracting $\bar{R}$ and squaring the result. The summation of the contributions from all independent variables computed in the same manner gives a close approximation of $U_{\bar{R}}$. In this case we replace the first order Taylor series analytic approximation, Equation 7, with the following finite difference approximation:

$$U_{\bar{R}} \approx \sqrt{[R(\bar{x}+U_{\bar{x}},\bar{y})-R(\bar{x},\bar{y})]^2+[R(\bar{x},\bar{y}+U_{\bar{y}})-R(\bar{x},\bar{y})]^2} \qquad (9)$$

This finite difference approximation may be much easier to evaluate. For example, the probable error , $U_{\bar{V}}$ of a cylinder is approximated as:

$$U_{\bar{V}} \approx \sqrt{[\pi(\bar{D}+U_{\bar{D}})^2\bar{L}/4-\pi\bar{D}^2\bar{L}/4]^2+[\pi\bar{D}^2(\bar{L}+U_{\bar{L}})/4-\pi\bar{D}^2\bar{L}/4]^2} \qquad (10)$$

An example of this method is also presented in the following Spreadsheet.

6. SPREADSHEET EXAMPLE

The file *Prop Errors Cylinder.xls* presents an experimental design example estimating the maximum probable error both analytically and numerically. The example also illustrates the use of a Visual Basic® function module. To see the function module, click the menu sequence *Tools*, *Macro* and *Visual Basic Editor*. Visual Basic® function modules are a useful tool for these calculations. The *Prop Errors Cylinder.xls* example can serve as a detailed guide on how to utilize spreadsheets for propagation of error analysis. The file *Prop Errors Cantilevered Beam.xls* presents an experimental design example with four independent variables.

7. RELATIVE UNCERTAINTIES

Sections 2, 3, 4, 5 and 6 above deal with absolute uncertainties. Equation 7 can be made non-dimensional by dividing both sides of the equation by R, squaring both sides and multiplying each term on the RHS by the appropriate factor $\left(\frac{\bar{x}}{\bar{x}}\right)^2=1$ or $\left(\frac{\bar{y}}{\bar{y}}\right)^2=1$, etc. The resulting equation is:

$$\frac{U_{\bar{R}}^2}{R^2} \approx \left(\frac{x}{R}\frac{\partial R}{\partial x}\bigg|_{\bar{x},\bar{y}}\right)^2\left(\frac{U_{\bar{x}}}{\bar{x}}\right)^2+\left(\frac{y}{R}\frac{\partial R}{\partial y}\bigg|_{\bar{x},\bar{y}}\right)^2\left(\frac{U_{\bar{y}}}{\bar{y}}\right)^2 \qquad (11)$$

The factors $\frac{U_{\bar{x}}}{\bar{x}}$ and $\frac{U_{\bar{y}}}{\bar{y}}$ are the relative uncertainties for the two independent variables, and these will generally be numbers much less than one. The factors that multiply the relative uncertainties,

$\left.\frac{x}{R}\frac{\partial R}{\partial x}\right|_{\bar{x},\bar{y}}$ and $\left.\frac{y}{R}\frac{\partial R}{\partial y}\right|_{\bar{x},\bar{y}}$, are called the uncertainty magnification factors (UMFs), and these factors indicate the influence of the uncertainty of a particular variable on the uncertainty of the result. A UMF value greater than 1 indicates the influence of the variable is magnified as it propagates through the result calculation equation. A UMF value of less than 1 indicates the influence diminishes as it propagates through the data equation into the result. The UMFs are particularly useful for identifying those factors that are most important in reducing the overall uncertainty. The uncertainties of the individual variables do not have to be known to analyze the UMFs.

Since the UMFs do not depend on the uncertainties of the variables, a second normalized form of Equation 7 is useful for finding the uncertainty percentage contributions (UPCs) from the variables to the uncertainty of the result squared. To obtain the UPCs divide both sides of Equation 7 by $U_{\bar{R}}$ and square both sides to give:

$$1 \approx \left(\left.\frac{\partial R}{\partial x}\right|_{\bar{x},\bar{y}}\right)^2 \frac{U_{\bar{x}}^2}{U_{\bar{R}}^2} + \left(\left.\frac{\partial R}{\partial y}\right|_{\bar{x},\bar{y}}\right)^2 \frac{U_{\bar{y}}^2}{U_{\bar{R}}^2} \qquad (12)$$

The UPC for the variable *x* is then defined as:

$$\mathrm{UPC}_{\mathrm{x}} = \frac{\left(\frac{\partial R}{\partial x}\right)^2 U_{\bar{x}}^2}{U_{\bar{R}}^2} \mathrm{x}\,100 = \frac{\left(\frac{\bar{x}}{R}\frac{\partial R}{\partial x}\right)^2 \left(\frac{U_{\bar{x}}}{\bar{x}}\right)^2}{\left(\frac{U_{\bar{R}}}{R}\right)^2} \mathrm{x}\,100 = \frac{(\mathrm{UMF})^2 \left(\frac{U_{\bar{x}}}{\bar{x}}\right)^2}{\left(\frac{U_{\bar{R}}}{R}\right)^2} \qquad (12)$$

The UPCs for the remaining variables have similar form.

These two non-dimension forms of the uncertainty equation are particularly useful for designing experiments to identify the major sources of error and to devise strategies to minimize the impact of the errors on the final result.

8. REFERENCES

[1] Coleman, Hugh W., Steele, W. Glenn, *Experimentation and Uncertainty Analysis for Engineers*, Second Edition, John Wiley & Sons, Inc., New York, 1999.

[2] International Organization for Standardization, *Guide to the Expression of Uncertainty in Measurement*, ISO, Geneva, 1993.

[3] *Physics 1210/1220 Lab Manual*, Department of Physics and Astronomy, University of Wyoming.

9. EXAMPLE PROPAGATION OF ERROR ANALYSIS

PROPAGATION OF ERROR ANALYSIS EXAMPLE

File: Prop_of_Errors.xls

The theoretical formula for the tip deflection of an end-loaded, circular cantilevered beam is:

$$y = \frac{64FL^3}{3\pi ED^4} \quad (1)$$

where: y = tip deflection (m)
F = tip loading (N)
L = length of rod (m)
E = Young's Modulus (Pa)
D = diameter of the rod (m)

Solving for Young's Modulus gives:

$$E = \frac{64FL^3}{3\pi yD^4} \quad (2)$$

Assuming:
– the rod is 1018 steel,
– Young's Modulus is about 197 Gpa,
– the minimum force is 15 N,
– the minimum rod length is 0.250 m and
– the minimum rod diameter is 0.005 m,
find the expected uncertainty for determining Young's Modulus by measuring the rod and the tip deflection.

Calculate the expected minimum deflection using Equation 1:

E =	1.97E+11	Pa	
F =	15	N	
L =	0.250	m	
D =	0.005	m	
y =	0.01293	m	$y=64*F*L^3/(3*PI()*E*D^4)$

Calculate the dimensionless UMFs for Equation 2:

UMF_F=	1.00	UMF_F= (F/E)(∂E/∂F) = F/E*64*L^3/(3*PI()*y*D^4)
UMF_L=	3.00	UMF_L= (L/E)(∂E/∂L) = L/E*64*F*L^2/(PI()*y*D^4)
UMF_D=	-4.00	UMF_D= (D/E)(∂E/∂D) = D/E*-256*F*L^3/(3*PI()*y*D^5)
UMF_y=	-1.00	UMF_y= (y/E)(∂E/∂y) = y/E*-64*F*L^3/(3*PI()*y^2*D^4)

$$\frac{\partial E}{\partial F} = \frac{64L^3}{3\pi yD^4}$$
$$\frac{\partial E}{\partial L} = \frac{64FL^2}{\pi yD^4}$$
$$\frac{\partial E}{\partial D} = \frac{-256FL^3}{3\pi yD^5}$$
$$\frac{\partial E}{\partial D} = \frac{-64FL^3}{3\pi y^2D^4}$$

Disregarding signs, the UMFs for the length and the diameter measurements indicate the uncertainties for these parameters will be magnified through the calculation of Young's Modulus.

The smallest measuring units for the instruments to be used in the experiment are:

smu_F =	0.5	N
smu_L =	0.01	m
smu_D =	0.001	m
smu_y =	0.002	m

Assuming a uniform distribution between graduations, the corresponding uncertainties are:

$t_{95\%}$ =	1.96	
U_F =	0.28	N
U_L =	0.006	m
U_D =	0.0006	m
U_y =	0.0011	m

Example propagation of error analysis (continued)

Calculate the expected uncertainty for the calculated Young's Modulus from Equation 2:

U_E = 9.19E+10 Pa

$U_E/E \times 100$ = 46.6%

Since the uncertainty of the result is about 50% of the magnitude of the estimated value, this experiment would have to be refined to achieve reasonable results.

Calculate the dimensionless UPCs for Equation 2 to find which variable(s) contribute the largest percentage of the errors:

UPC_F=	0.2 %
UPC_L=	2.1 %
UPC_D=	94.2 %
UPC_y=	3.5 %
	100 %

For this experiment the diameter contributes the majority of the uncertainty.