HW3

(Due Tuesday, October 29, 2013)

This optional HW2a is an opportunity for students wishing to improve their HW2 grade.

1. Let \( \tau = \frac{1 + \sqrt{5}}{2} \).

   (a) Show that \( \tau^2 = \tau + 1 \).

   (b) Consider \( \mathbb{Z}[\tau] = \{a + b\tau : a, b \in \mathbb{Z}\} \). Using (a), show that \( R \) is a commutative ring with identity.

   (c) Expand each of the following ring elements, and express in the standard form \( a + b\tau \) where \( a, b \in \mathbb{Z} \):
      
      - \((7 + 4\tau)(3 - 5\tau)\)
      - \(\tau^3\)
      - \(\tau^4\)
      - \(\tau^5\)

   (d) Find a formula expressing \( \tau^n \) in the standard form \( a + b\tau \) where the integer coefficients \( a, b \) are given in terms of a well-known sequence. Prove your formula by induction.

2. Let \( R \) be the set of all 2 matrices with elements in \( \mathbb{Z}_2 \). Addition and multiplication of \( 2 \times 2 \) matrices are defined as usual; but since entries are in \( \mathbb{Z}_2 \), they are added and multiplied as integers modulo 2).

   (a) Is \( R \) commutative? Justify your answer.

   (b) What is the identity element of \( R \)?

   (c) How many elements does \( R \) have?

   (d) How many units does \( R \) have? Which elements of \( R \) are units?

3. (a) How many elements \( z \in \mathbb{Z}[i] \) are there satisfying \( |z|^2 = 101 \)? List them. Which of these elements are irreducible? (Here \( i = \sqrt{-1} \).

   (b) How many elements \( z \in \mathbb{Z}[\sqrt{-2}] \) are there satisfying \( |z|^2 = 101 \)? List them. Which of these elements are irreducible?
4. Factor 30 into irreducible factors
   (a) in \( \mathbb{Z} \);
   (b) in \( \mathbb{Z}[\sqrt{-2}] \);
   (c) in \( \mathbb{Z}[\sqrt{3}] \);
   (d) in \( \mathbb{Z}[i] \) where \( i = \sqrt{-1} \);
   (e) in \( \mathbb{Z}[\sqrt{7}] \).

   You may use any facts stated (with or without proof) in the handout on Factorization in Rings.

5. Prove that the ring \( \mathbb{Z}[\sqrt{-6}] \) does not have unique factorization.

   \textit{Hint:} Imitate the proof, given in class, that \( \mathbb{Z}[\sqrt{-5}] \) does not have unique factorization.