Sample Test 2
November, 2013

This sample test is intended to resemble the Second Test (Tuesday, November 26, 2013 during class time) in approximate length, difficulty, and style, although clearly the content may differ. The actual content will consist of all material covered in class since Test 1, and all related handouts.

**Instructions.** The only aids allowed are a hand-held calculator and one ‘cheat sheet’, i.e. an 8.5”×11” sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Time permitted: 75 minutes.

1. **(18 points)** In each case, you are given a map \( \theta \) between rings. State whether or not \( \theta \) is a homomorphism of rings. If \( \theta \) is a homomorphism, no justification is required; but if \( \theta \) is not a homomorphism, justify this assertion.
   (a) \( \theta : \mathbb{Z}[t] \to \mathbb{Z}[t] \) defined by \( f(t) \mapsto f(t^2) \).
   (b) \( \theta : M_2(\mathbb{R}) \to M_2(\mathbb{R}) \) defined as the ‘transpose map’ \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix} \).
   (c) \( \theta : F \to F, \ x \mapsto x^2 \) where \( F = \{0, 1, \alpha, \beta\} \) is the field with four elements.
   (d) \( \theta : \mathbb{R} \to \mathbb{C}, \ x \mapsto x \).
   (e) \( \theta : M_2(\mathbb{R}) \to \mathbb{R}, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d \).
   (f) \( \theta : \mathbb{Z}[t] \to \mathbb{R}, \ f(t) \mapsto f(\pi) \) where \( \pi = 3.141592653 \ldots \) as usual.

2. **(10 points)** Give an example of a ring \( R \) and a subring \( S \subseteq R \) which is *not* an ideal. Justify your answer.

3. **(12 points)** Let \( R = \mathbb{Z}[t]/\langle t^2-t-1 \rangle \), and let \( \alpha = t + \langle t^2-t-1 \rangle \in R \). Express each of the following elements of \( R \) in standard form as \( \Box \ t + \Box + \langle t^2-t-1 \rangle \) where each ‘\( \Box \)’ is an integer.
   (a) \( \alpha^2 \)
   (b) \( \alpha^3 \)
   (c) \( \alpha^4 \)
4. (15 points) Let $\theta : \mathbb{Z} \to \mathbb{Z}$ be a homomorphism of rings. Show that either $\theta$ is the trivial map $x \mapsto 0$ for all $x \in \mathbb{Z}$, or $\theta$ is the identity map $x \mapsto x$ for all $x \in \mathbb{Z}$.

5. (15 points) Let $p(x) = x^3 - x - 1$. Show that $p(x)$ is irreducible in $\mathbb{Q}[x]$. (Hint: Suppose $\frac{a}{b} \in \mathbb{Q}$ is a root of $p(x)$. We may assume the fraction $\frac{a}{b}$ is in lowest terms, i.e. $a, b$ are relatively prime integers. Proceed to find a contradiction.)

6. (30 points) Answer TRUE or FALSE to each of the following statements.

(a) Every integral domain has unique factorization.  

(b) Every integral domain is a subring of a field. 

(c) Every subring of a field is an integral domain. 

(d) If $u \in R$ where $R$ is a commutative ring with identity, and $u$ is a unit, then $u$ is not a zero divisor. 

(e) If $u \in R$ where $R$ is a commutative ring with identity, and $u$ is not a zero divisor, then $u$ is a unit. 

(f) Every field $F$ has exactly two ideals, the trivial ideal $\{0\}$ and $F$ itself. 

(g) If $R$ is an integral domain, then every polynomial $f(t) \in R[t]$ of degree $n \geq 1$ has at most $n$ roots. 

(h) In the ring $\mathbb{Z}[t]$, every ideal is principal. 

(i) If $R$ is a ring with identity 1, then there is exactly one ideal containing 1. 

(j) Let $F$ be a field, and let $p(t) \in F[t]$ be an irreducible polynomial. If $u(t), v(t) \in \langle p(t) \rangle$ then either $u(t) \in \langle p(t) \rangle$ or $v(t) \in \langle p(t) \rangle$. 

_____ (True/False)