#1. \( H = 2 \cdot 7 = (1 + \sqrt{13})(1 - \sqrt{13}) \). We will show that these are essentially distinct factorizations into irreducibles. First observe that \( \mathbb{Z}[\sqrt{13}]^* = \{1, -1\} \) since the only solutions of \( N(a + b\sqrt{13}) = a^2 + 13b^2 = 1 \) are \((a, b) = (\pm 1, 0) \).

To show that 2 is irreducible in \( \mathbb{Z}[\sqrt{13}] \), suppose \( 2 = xy \), \( x, y \in \mathbb{Z}[\sqrt{13}] \) with neither factor \( x, y \) a unit. Then \( 4 = N(2) = N(x)N(y) \) so \( N(x) = N(y) = 2 \). However, \( N(a + b\sqrt{13}) = a^2 + 13b^2 = 2 \) has no solutions with \( a, b \in \mathbb{Z} \), a contradiction. Thus, 2 is irreducible. Similar reasoning shows that 7, \( 1 + \sqrt{13} \), \( 1 - \sqrt{13} \) are irreducible.

Finally, the factors \( 1 \pm \sqrt{13} \) are not associates of the factors 2, 7 as is easily seen since the only units are 1, -1.

#2. (a) This is not a ring; it does not contain a zero element (additive identity).

(b) This is not a ring; the associative law fails since \((i \times i) \times j = 0 \times j = 0 \) whereas \( i \times (i \times j) = i \times k = -j \).

(c) This is a ring (a subring of \( \mathbb{M}_2(\mathbb{R}) \) since it contains \( \mathbb{R} \)) and is closed under addition, subtraction and multiplication.

(d) This is not a ring since it is not closed under multiplication: \((0 1)(0 1) = (0 0)\).

(e) This is not a ring since the distributive law fails. Let \( f(t) = t \); then \((f * (f + f))(t) = t \) whereas \((f * f)(t) + (f * f)(t) = t + t = 2t \).
3. (a) Suppose \( f(x) \) has a rational root \( \frac{a}{b} \) where \( a, b \in \mathbb{Z} \) with \( \gcd(a, b) = 1 \). Then \( a^3 - ab^2 + 2b^3 = 0 \). If \( b \) has a prime divisor \( p \), then \( p \mid a^3 \) so \( \frac{a}{b} \), a contradiction, thus \( b = \pm 1 \). If \( a \) has a prime divisor \( p \), then \( p \mid 2 \) so \( p = 2 \); thus, \( \frac{a}{b} \in \{-2, -1, 0, 1, 2\} \). However, we readily check that none of \(-2, -1, 0, 1, 2\) is a root of \( f(x) \). So \( f(x) \) has no root in \( \mathbb{Q} \), which means \( f(x) \) has no factor of degree 1. Since \( \deg f(x) = 3 \), it follows that \( f(x) \) is irreducible in \( \mathbb{Q}[x] \).

(b) Take \( F = \mathbb{Q}[x]/\langle f(x) \rangle \). Every element of \( F \) has the form \( a + bx + cx^2 + \langle f(x) \rangle \), which we abbreviate as \( a + bx + cx^2 \). Here \( \alpha \in F \) satisfies \( f(\alpha) = 0 \).

(c) The relation \( \alpha^3 - \alpha = -2 \) may be factored as \( (x+1)(\alpha^2 - \alpha) = -2 \), so \( \frac{\alpha}{\alpha+1} = \frac{1}{2} (\alpha - \alpha^2) = 0 + \frac{1}{2} \alpha - \frac{1}{2} \alpha^2 \).

Alternatively, \( \gcd(x+1, x^3-x+2) = 1 \) in \( \mathbb{Q}[x] \) since \( x^3-x+2 \) is irreducible. Proceeding with the extended Euclidean algorithm:

\[
\begin{array}{c|ccc}
\text{x}^3-x+2 & x+1 \\
\hline
1 & 0 & x^3-x+2 \\
0 & 1 & x+1 \\
1 & -x^2+x & 2 \\
\frac{1}{2} & -\frac{1}{2} x^2+\frac{1}{2} x & 1 \\
\end{array}
\]

i.e. \( 1 = \frac{1}{2} (x^3-x+2) + (x+1) \left( \frac{1}{2} x - \frac{1}{2} \alpha^2 \right) \)

so \( (x+1 + \langle f(x) \rangle) \left( \frac{1}{2} x - \frac{1}{2} \alpha^2 + \langle f(x) \rangle \right) = 1 + \langle f(x) \rangle \) in \( \mathbb{Q}[x] \). Passing to the quotient ring \( F = \mathbb{Q}[x]/\langle f(x) \rangle \) we obtain \( (x+1) \left( \frac{1}{2} x - \frac{1}{2} \alpha^2 \right) = 1 \), i.e. \( \frac{1}{\alpha+1} = \frac{1}{2} \alpha - \frac{1}{2} \alpha^2 \).
#4. (a) Multiplying both sides of \( 1+1 = 0 \) by \( x \in \mathbb{R} \) gives \( 2x = x + x = 0 \) for all \( x \in \mathbb{R} \). Now
\[
\theta (x+y) = (x+y)^2 = x^2 + 2xy + y^2 = x^2 + y^2 = \theta (x) + \theta (y)
\]
and \( \theta (xy) = (xy)^2 = x^2 y^2 = \theta (x) \theta (y) \) for all \( x, y \in \mathbb{R} \).

(b) Examples include \( \mathbb{Z}_2 [x] \) and the field of elements studied in class.

#5. The evaluation map \( \theta : \mathbb{R} \to \mathbb{R} \) given by \( f \mapsto f(0) \)
is (as always) a homomorphism. It is onto \( \mathbb{R} \) since constant functions map onto \( \mathbb{R} \).

#6. Recall that every ideal of \( \mathbb{Z} \) is principal.
(a) The only such ideals are \( \langle p \rangle \) and \( \mathbb{Z} \) itself.
This is because any such ideal has the form \( \langle m \rangle \) with \( m \in \mathbb{Z} \); and \( pq, pr \in \langle m \rangle \) requires \( m | pq \) and \( m | pr \), so \( m \) divides \( \gcd (pq, pr) = p \).
This forces \( \langle m \rangle = \langle 1 \rangle = \mathbb{Z} \) or \( \langle m \rangle = \langle p \rangle \).
(b) Of the two ideals listed in (a), only \( \mathbb{Z} \) is possible since \( q \not\in \langle p \rangle \).

#7. (a) F (b) T (c) F (d) T (e) F (f) T (g) F (h) T (i) T (j) T

Remarks:
(a) \( \mathbb{Z}[\sqrt{2}] \) has unique factorization, as mentioned in class. The factors 2, 7 and \( 6 + 2\sqrt{2} \) are reducible:
\[ 7 = (3 + \sqrt{2})(3 - \sqrt{2}), \quad 2 = (\sqrt{2})(\sqrt{2}), \quad 6 + 2\sqrt{2} = (\sqrt{2})(\sqrt{2})(3 + \sqrt{2}) \]
(b) \( f(x) \mapsto f(t) \) is an isomorphism \( \mathbb{Z}[x] \to \mathbb{Z}[t] \).
(c) \( xf(x) = 1 \) has no solution \( f(x) \in \mathbb{Z}[x] \) since if \( f(x) \neq 0 \) then \( \deg (xf(x)) > 1 \).
(d) \( 3 \cdot 17 = 0 \) in \( \mathbb{Z}_5 \).

(e) \( x^4 + 1 \in \mathbb{R}[x] \) has no real roots but it is reducible in \( \mathbb{R}[x] \): 
\[ x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1). \]

(f) This follows from Euclid's Algorithm since \( \mathbb{R} \) is a field.

(g) \( F[[t]] \) is a field; its only ideals are \{0\} and \( F[[t]] \) itself. \( F[t] \) is however a subring of \( F[[t]] \).

(h) \( a^2 = a \cdot a \) which uses only multiplication (which is well-defined in \( \mathbb{Z}_n \)).

(i) \((x + J)(y + J) = xy + J = yx + J = (y + J)(x + J)\).

(j) \((x + J)(1 + J) = xJ + J = x + J = 1x + J = (1 + J)(x + J).\)