#1. (a) $\mathbb{Z}_{20}$ (with the usual addition and multiplication mod 20).
(b) $R$ is the set of real numbers with the usual addition, but with a new multiplication $xy = 0$ for all $x, y \in R$.
(c) The even integers, with the usual addition and multiplication.
(d) $M_2(\mathbb{R})$, the set of all $2 \times 2$ real matrices. (Use the usual addition and matrix multiplication.)
(e) $M_2(\mathbb{Z})$, the set of all $2 \times 2$ matrices with even integer entries, with the usual addition and multiplication of matrices.
(f) $R = \{0, a, b, c, d\}$ with the binary operations:

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<th>a</th>
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<tr>
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(Note that $R$ is a field.)

Or: replace this multiplication with the multiplication where $xy = 0$ for all $x, y \in R$.

#2. (a) Yes, $R$ is commutative:

$$(x, y)(x', y') = (xx', xy' + xy, x'y') = (x, y')(x', y)$$

for all $x, y, x', y' \in R$.

(b) The identity is $(1, 0)$:

$$(x, y)(1, 0) = (x'1, x0 + y'0) = (x, y)$$

for all $x, y \in R$.

(c) In order for $(x, y)(u, v) = (1, 0)$, we must have

$$xu = 1, xv + yu = 0.$$ 

This has a solution for $(u, v) \in R$, given $(x, y) \in R$, iff $x \neq 0$, in which case...
\[(x,y)^{-1} = (u,v) = \left(x^{-1}, -yx^{-2}\right) = (\frac{1}{x}, -\frac{y}{x^2}).\]

The units of \(R\) are \(R^* = \{ (x,y) \in R : x \neq 0 \}\).

(d) \(R \neq C\) since \(C\) is a field (every nonzero element of \(C\) is a unit) but this is not true of \(R\).

Also, \(R \neq R^2\) where \(R^2\) has coordinatewise addition and multiplication. In \(R\) we have
\[(0,1)^2 = (0,1)(0,1) = (0,0)\]
but in \(R^2\), the square of every nonzero element is nonzero: if \((x,y) \neq (0,0)\) in \(R^2\), then
\[(x,y)^2 = (x,y)(x,y) = (x^2, y^2) \neq (0,0)\] in \(R^2\).

#3. Suppose \(x^2 = x\) for every \(x \in R\). Then
\[(x+y)^2 = x^2 + xy + yx + y^2 = x + y = x + y^2\]
so \(xy + yx = 0\) for all \(xy \in R\). In particular \(x + x = x^2 + x^2 = 0\) for all \(x \in R\) (this follows from the previous identity as a special case when \(y = x\)). Thus \(-x = x\) for all \(x \in R\) and
\[yx = -xy = (-x)y = xy.\]

#4. (a) Every rational number of the form \(\pm \frac{1}{2^k}\) (where \(k \in \mathbb{Z}\)) is a unit since \((\pm \frac{1}{2^k})(\pm \frac{1}{2^k}) = 1\).

These are the only units since if \(\frac{a}{2^k} \cdot \frac{b}{2^k} = 1\) where \(a, b \in \mathbb{Z}\) then by considering the prime factorizations of \(a\) and \(b\), \(\text{labl} = \text{lal} \cdot \text{lbl} = 1\) must be a power of 2.

(b) Numbers of the form \(\pm \frac{p}{2^k}\) are irreducible where \(k \in \mathbb{Z}\) and \(p\) is any odd prime. (These are the only irreducibles.)
Comments:

(b) \( \mathbb{Z}[\sqrt{5}] \) does not have unique factorization, as shown in class.

(c) \( \mathbb{Z}[i] \) has only 4 units \((\pm 1, \pm i)\).

(d) \( \mathbb{Z}[\sqrt{2}] \) has only 2 units \((\pm 1)\).

(e) \( 29 = (2+5i)(2-5i) \)

(f) If \( 31 = xy \) then \( 31^2 = N(x)N(y) \), but \( N(x) = 31 \) has no solutions, so one of \( N(x) \) or \( N(y) \) is 1, and then \( x \) or \( y \) is a unit.

(g) If \( uu^{-1} = u' u = 1 \) and \( vv^{-1} = v' v = 1 \), then \((uv)(v^{-1}u^{-1}) = 1 \) and \((v^{-1}u^{-1})(uv) = 1 \), so \((uv) = v' u'\).

(h) In \( \mathbb{Z}_{24} \), the equation \( x^2 = 1 \) has eight solutions: all units i.e. \( 1, 5, 7, 11, 13, 17, 19, 23 \).

(i) The fact that \( x^2 x = xx^2 \) follows from associativity.