1. No, \( R \not\cong C \). As evidence for this, it suffices to say that \( C \) has a root of \( x^2 + 1 = 0 \) but \( R \) doesn’t.

To express this more formally, suppose \( \Theta: C \to R \) is an isomorphism. As explained in class, \( \Theta(0) = 0 \) and \( \Theta(i) = 1 \); this follows from the uniqueness of 0 and 1, as identities for addition and multiplication respectively. Then

\[
0 = \Theta(0) = \Theta(i^2 + i) = \Theta(i^2) + \Theta(i) = \Theta(i)^2 + 1
\]

so \( \Theta(i) \in R \) is a root of \( x^2 + 1 = 0 \), a contradiction.

2. (a) No, it is not a ring; most elements have no additive inverse (e.g. \( 5 + x = 0 \) has no solution).

(b) \( U \) is not a ring; it has no zero element i.e. no element \( x \) satisfying \( x + z = x \) for all \( x \in U \).

(c) \( S \) is not a ring; it has no zero element i.e. no element \( z \) satisfying \( x + z = x \) for all \( x \in S \).

(d) \( R \) is a ring (commutative, with zero element \( -1 \) and identity 0 i.e. \( x + 0 \cdot (-1) = x \), \( x \cdot 0 = x \) for all \( x \in R \)). If \( Z \) is the usual ring of integers then \( \Theta: R \to Z \) given by \( \Theta(r) = r + 1 \) is an isomorphism, so \( R \cong Z \).

(e) \( T \) is not a ring; for example it fails the left-distributive law. If \( f(x) = x^2 \), \( g(x) = x \), \( h(x) = 1 \) then \( f(1) = (g + h) \neq (f \circ g) + (f \circ h) \) since \((f \circ (g + h))(x) = (x + 1)^2 \), \((f \circ g)(x) = x^2 \), \((f \circ h)(1) = 1 \).
#3. Units in $\mathbb{Z}_{18}$ are represented by integers $a \in \{0, 1, 2, \ldots, 17\}$ such that $\text{gcd}(a, 18) = 1$, so there are six units $1, 5, 7, 11, 13, 17$.

\[
\begin{array}{cccccc}
   \text{a} & 1 & 5 & 7 & 11 & 13 & 17 \\
   \text{a'} & 1 & 11 & 13 & 5 & 7 & 17 \\
\end{array}
\]

#4. (a) Units in $R$ are fractions of the form $\frac{a}{b}$ where $a, b$ are odd integers.

This is because an element $\frac{a}{b} \in R$ (with $a \in \mathbb{Z}$ and $b$ odd) is a unit iff its inverse $\frac{b}{a}$ is also in $R$; this requires $a \in \mathbb{Z}$ to be odd also.

(b). $2 = \frac{2}{1} \in R$ is irreducible; more generally, $\frac{2a}{b} \in R$ (where $a, b$ are odd) is irreducible. These are all the irreducible elements of $R$.

And by (a), they are all associates of each other.

To see this, suppose $\frac{a}{b} \in R$ is irreducible where $a \in \mathbb{Z}$ and $b$ is odd. Since $\frac{a}{b}$ is not a unit (by definition), $a = 2c$ for some $c \in \mathbb{Z}$ and $c \neq 0$. If $c$ is even then $c = 2d$ and we have $\frac{a}{b} = 2 \cdot \frac{2d}{b}$ where neither of the factors $2, \frac{2d}{b}$ is a unit. This cannot happen, so $c$ is odd.
5. (a) T  (b) F  (c) F  (d) T  (e) T  (f) F  (g) T  (h) T  (i) T  (j) T

Comments:

(a) \( \mathbb{Z}_p \) is a commutative ring with identity. If \( a \in \{1, 2, 3, \ldots, p-1\} \) then \( \gcd(a, p) = 1 \) so \( ax + py = 1 \) for some \( x, y \in \mathbb{Z} \) by the extended Euclidean algorithm, so \( ax \equiv 1 \mod p \); hence \( a \) represents a unit in \( \mathbb{Z}_p \).

(c) The only units in \( \mathbb{Z}[x] \) are the two constant polynomials \( 1 \) and \( -1 \).

(d) Units in \( \mathbb{Z}[12] \) are \( \pm (1+\sqrt{2})^k \), \( k \in \mathbb{Z} \).

(b) \((x+y)^2 = (x+y)(x+y) = x^2 + xy + yx + y^2\). This doesn't simplify any further unless \( xy = yx \), which only works if \( R \) is commutative.

(e) If \( 23 = xy \) where \( x, y \in \mathbb{Z}[i] \) then \( 23 = N(x)N(y) \).

We cannot have \( N(x) = 23 \) since \( a^2 + b^2 = 23 \) has no integer solutions; so \( N(x) = 1 \) or \( N(y) = 1 \), i.e. either \( x \) or \( y \) is a unit.

(f) \( 53 = (7+2i)(7-2i) \) where neither \( 7+2i \) nor \( 7-2i \) is a unit.

(g) If \( uv = 1 \) then \((-u)(-v) = 1\).

(h) As indicated in class, for large integers in general, factorization is prohibitively difficult; however, Euclid's algorithm is very efficient.

(i) If \( x, y \in S \) then \( (x+y)r = x(r)+yr = x+ry = r(xy) \)
so \( x+y \in S \); also \( (xy)r = x yr = x(ry) = x(r)y = (xr)y = (rx)y = r(xy) \)
so \( xy \in S \). Thus \( S \) is closed under addition and multiplication. The other requirements are consequence of the fact that \( R \) is a ring.

(j) See class notes or the handout in the Fund. Theorem of Arithmetic.