Math 3700

Combinatorics

Book 2
Stirling numbers

Count the number of functions $[n] \to [x]$ (or from any $n$-set to any $x$-set):

$x^n = \sum_{k=0}^{\lfloor x \rfloor} S(n,k) (x)_k$

If $k$ is the size of the range $k = |\{ f(i) : i \in [n] \}|$

we can count the number of functions with each possible size $k$ for the range as follows:

There are $S(n,k)$ partitions of $[n]$ which prescribe

the subset of the domain mapping to the first element of the range;

Then we can choose the first element of the range in $x$ ways;

There are $x(x-1)(x-2) \cdots (x-k+1) = (x)_k$ lists of length $k$ with distinct elements

that can be chosen from $[x]$.

Eg, $x^3 = S(3,1)(x) + S(3,2)(x)_2 + S(3,3)x_3$ is the number of functions $[3] \to [x]$

check:

$x + 3x^2 - 8x + x^3 - 3x^2 + 2x = x^3$. 

The Bell numbers \( B(n) \) = no. of partitions of \([n]\) into an arbitrary number of subsets satisfy a recurrence formula \[ B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i) \]

\( B(0) = 1 \)
\( B(1) = 1 \)

\( B(2) = 1 \cdot B(0) + 1 \cdot B(1) = 1 + 1 = 2 \)
\( B(3) = 1 \cdot B(0) + 2 \cdot B(1) + 1 \cdot B(2) = 1 + 2 + 2 = 5 \)
\( B(4) = 1 \cdot B(0) + 3 \cdot B(1) + 3 \cdot B(2) + 1 \cdot B(3) = 1 + 3 + 6 + 5 = 15 \)

\[ f(x) = \sum_{n=0}^{\infty} \frac{B(n)}{n!} x^n \] is the exponential generating function for the sequence of Bell numbers.

**Theorem:** \[ f(x) = \sum_{n=0}^{\infty} \frac{B(n)}{n!} x^n = e^{e^x-1} \]

**Proof:** Idea: \( g(x) = e^{e^x-1} \) satisfies \[ g'(x) = e^{e^x} \cdot e^x = e^x g(x) \]

Show that \( f \) satisfies the same differential equation.
\[ f(x) = \sum_{n=0}^{\infty} \frac{B(n)}{n!} x^n \]

\[ f'(x) = \sum_{n=0}^{\infty} \frac{B(n)}{n!} x^{n-1} = \sum_{n=1}^{\infty} \frac{B(n)}{(n-1)!} x^{n-1} = \sum_{m=0}^{\infty} \frac{B(m+1)}{m!} x^m \]

\( (m = n - 1, \quad n = m + 1) \)

\[ \sum_{m=0}^{\infty} \sum_{i=0}^{m} \frac{1}{m!} \binom{m}{i} B(i) x^m = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{x^m}{(m-1)!} B(i) x^m \]

\[ \sum_{m=0}^{\infty} \sum_{i=0}^{m} \frac{x^{m-i}}{(m-1)!} \frac{B(i)}{i!} x^i \]

\[ \begin{cases} j = m - i & \leftrightarrow \begin{cases} m = j + i \\ 0 \leq i < \infty \\ 0 \leq j < \infty \end{cases} \\ 0 \leq i \leq m < \infty \end{cases} \]

\[ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^j}{j!} \frac{B(i)}{i!} x^i = e^x f(x) \]

\[ \frac{d}{dx} (\ln f(x)) = \frac{1}{f(x)} f'(x) = \frac{f'(x)}{f(x)} = e^x = \frac{d}{dx} (e^x) \]

\[ \frac{d}{dx} (\ln f(x) - e^x) = e^x - e^x = 0 \]

\[ \ln f(x) - e^x = c = \text{constant} \]

\[ \ln f(x) = e^x + c \]

\[ \ln f(0) = 1 + c \]

\[ 0 = 1 + c \Rightarrow c = -1 \]

\[ \ln f(x) = e^x - 1 \Rightarrow f(x) = e^{e^x - 1} \]
How many ways can we parenthesize a product of \( n \) symbols? 

\[
(x y) z = x (y z) \\
(x y) (z w) = ((x y) z) w = (x (y z)) w = (x y (z w)) = (x (y (z w))) = (x (y z)) w
\]

\[
\begin{array}{c|c}
\text{n: no. of symbols} & \text{no. of ways to parenthesize} \\
\hline
0 & 1 \\
1 & 1 \\
2 & 1 \\
3 & 2 \\
4 & 5 \\
5 & 14 \\
6 & \\
\end{array}
\]

\[
uvwxy : \\
\begin{align*}
((uv)w)x & y \\
((u(v)w)x & y \\
(u(v)(wx))y & \\
(u (v (wx)))y & \\
(u((vw)x))y & \\
(u((vw)(xy)))y & \\
\end{align*}
\]
<table>
<thead>
<tr>
<th>n</th>
<th>No. Ways</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>19</td>
</tr>
<tr>
<td>5</td>
<td>?</td>
</tr>
<tr>
<td>6</td>
<td>?</td>
</tr>
</tbody>
</table>
Let \( C_n \) be the number of shortest paths (i.e., paths of length \( 2n \)) from (0,0) to (n,n) using \( n \) steps east (i.e., adding 1 in the x direction) and \( n \) steps north (i.e., adding 1 in the y direction) without going above the line \( y = x \).

Remark: Without the last constant, there are \( \binom{2n}{n} \) paths from (0,0) to (n,n).

For \( n = 2 \):

\[
\text{EENN, ENEN, ENNE, NEEN, NNEE}
\]

There are \( \binom{2n}{n} \) words of length \( 2n \) using an alphabet of two symbols N, E having \( n \) N's and \( n \) E's.

Only two of these (EENN and ENEN) stay below the line \( y = x \).

Before finding an explicit formula for \( C_n \), we first derive a recursive formula (recurrence formula).

Every shortest path comes back to the line \( y = x \) at a point (k,k) with \( 1 \leq k \leq n-1 \) with \( k \) being the smallest such positive number (i.e., (k,k) is the first point after (0,0) where we come back to the line \( y = x \)). For each \( k \in \{1, 2, \ldots, n\} \) we consider the number of shortest paths:
\[ C_n = \sum_{k=0}^{n} C_{k-1} C_{n-k} \]

Catalan numbers: \( 1, 1, 2, 5, 14, \ldots \)

Wednesday: explicit formula for \( C_n \). (\( C_0, C_1, C_2, C_3, \ldots \) is the sequence of Catalan numbers.)
There are \((\binom{2n}{n})\) shortest paths (i.e., paths of length \(2n\)) along an \(n \times n\) grid, from \((0, 0)\) to \((n, n)\). Each such path is specified as a string of length \(2n\) over the alphabet \(\{E, N\}\).

The generating function for \(\binom{2n}{n}\) is \(\frac{1}{\sqrt{1 - 4x}}\). 

**Proof:**

\[
\frac{1}{\sqrt{1 - 4x}} = (1 - 4x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \left(\frac{\binom{1/2}{n}}{n!}\right)(-4x)^n \quad \text{(Binomial Theorem)}
\]

\[
\binom{1/2}{n} = \frac{\Gamma(1/2 + n)}{\Gamma(1/2 + 1)n!}
\]

\[
= \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \cdot \frac{1}{2^n} \cdot \frac{1}{\sqrt{\pi}} \xrightarrow{n \to \infty} \frac{x(x-1)(x-2)\cdots(x-n+1)}{n(n-1)(n-2)\cdots1} = \frac{(x)_n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \frac{(2n)!}{n! n!} x^n = \sum_{n=0}^{\infty} \binom{2n}{n} x^n.
\]
Define $C(x) = C_0 + C_1 x + C_2 x^2 + \cdots = \sum_{n=0}^{\infty} C_n x^n$ (the generating function for the sequence of Catalan numbers). Obtain an explicit formula for $C(x)$ using the recurrence formula, then read off coefficients in the power series for $C(x)$ to get $C_n$.

$$C(x) = \sum_{n=0}^{\infty} C_n x^n = 1 + \sum_{n=1}^{\infty} C_n x^n = 1 + \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} C_{k-1} C_{n-k} \right) x^n$$

$$= 1 + x \sum_{n=1}^{\infty} \sum_{k=1}^{n} C_{k-1} C_{n-k} x^{k-1} C_{n-k} x^{n-k}$$

$$= 1 + x \sum_{k=1}^{\infty} C_{k-1} x^{k-1} \sum_{n=k}^{\infty} C_{n-k} x^{n-k}$$

$$= 1 + x \left( \sum_{r=0}^{\infty} C_r x^r \right) \left( \sum_{s=0}^{\infty} C_s x^s \right)$$

$$= 1 + x \left( \sum_{r=0}^{\infty} C_r x^r \right)$$

$$x C(x) - C(x) + 1 = 0 \Rightarrow C(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x} = \frac{1 \pm (1 - 2x - 2x^2 - \cdots)}{2x}$$

We must choose the $\frac{1 \pm \sqrt{1 - 4x}}{2x}$ in order to obtain non-negative coefficients.

$$C(x) = \frac{1 - (1 - 2x - 2x^2 - 4x^3 - 10x^4 - 28x^5 - \cdots)}{2x} = \frac{2x + 2x^3 + 4x^5 + 10x^7 + 28x^9 + \cdots}{2x}$$

$$= \frac{1 + x + 2x^2 + 5x^3 + 14x^4 + \cdots}{2x}$$
\[
C(n) = \frac{1}{2^n} \left( 1 - \sqrt{1 - 4x} \right) = \frac{1}{2^n} \left( 1 - \sum_{n=0}^{\infty} \left( \frac{1}{n} \right)^n (-4x)^n \right) \\
= \frac{1}{2x} \left( 1 - 1 - \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)^n (-4x)^n \right) \\
= -\frac{1}{2x} \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)^n (-4x)^n \\
= -\frac{1}{2x} \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{1}{2} \right)\left( \frac{-1}{2} \right)\left( \frac{-3}{2} \right) \cdots \left( \frac{-(2n-3)/2}{2} \right) (-4x)^n \\
= \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{n!} \frac{2^{n-1}}{n} x^{n-1} \quad \text{for} \quad k = n-1, \quad n = k+1 \\
= \sum_{k=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{(k+1)!} \frac{2^k}{k!} x^k \\
= \sum_{k=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{k!} \frac{2^k}{k!} x^k = \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} x^k \\
\text{so} \quad C_k = \frac{1}{k+1} \binom{2k}{k} \\
C_4 = \frac{1}{5} \cdot \binom{8}{4} = \frac{1}{5} \cdot \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 14
Partitions of $n$ and partitions of $[n] = \{1, 2, \cdots, n\}$.

Eg. $n=5$: $\begin{align*}
5 &= 5 \quad p_1(5) = 1 \\
   &= 4+1 \quad p_2(5) = 2 \\
   &= 3+2 \quad p_3(5) = 2 \\
   &= 3+1+1 \quad p_4(5) = 2 \\
   &= 2+2+1 \quad p_5(5) = 2 \\
   &= 2+1+1+1 \quad p_6(5) = 1 \\
   &= 1+1+1+1+1 \quad p_7(5) = 1
\end{align*}$

There are $p(5) = 7$ partitions of 5

(1 partition into 1 part, 2 parts, 3 parts, etc.)

Partitions of $[5] = \{1, 2, 3, 4, 5\}$

Eg. partitions of type $(3, 1, 1)$ i.e. $\{*, *, *, 3, 1, 1\}$: $(3, 1, 1) = \frac{5!}{3!1!1!} = \frac{120}{6} = 20 \div 2 = 10$

(20 = $\binom{5}{3}$, 1 way to choose a list $\{*, *, *, 3, 1, 1\}$ then divide by 2 since the order of the two subsets of size 1 is irrelevant)

$\begin{align*}
\{1, 2, 3, 4, 5\} \\
\{1, 2, 5\}, \{1, 3, 3\}, \{1, 4, 4\} \\
\{1, 3, 5\}, \{1, 2, 4, 3\} \\
\{1, 3, 6\}, \{1, 2, 5, 3\} \\
\{1, 3, 7\}, \{1, 2, 6, 3\} \\
\{1, 3, 8\}, \{1, 2, 7, 3\}
\end{align*}$
Theorem 5.22 (p. 102)

The number of partitions of \([n]\) of type \((a_1, a_2, \ldots, a_k)\) is

\[
\left( \begin{array}{c} n \\ a_1, a_2, \ldots, a_k \end{array} \right) \quad \text{no. of lists } A_1, A_2, \ldots, A_k \quad \prod m_i!
\]

where \(m_i, m_2, \ldots\) are the number of repetitions of terms in \((a_1, a_2, \ldots, a_k)\),

\( (a_1, \ldots, a_k) \) is a partition of \(n \) of \( n = a_1 + a_2 + \cdots + a_k \)
\( a_1 > a_2 > \cdots > a_k > 1 \) pos. integers

\(|A_i| = a_i \)

\( A_1, \ldots, A_k \) partition of \([n]\) (order matters in the list)
In [5], the no. of partitions of each type:

<table>
<thead>
<tr>
<th>Type</th>
<th>Expression</th>
<th>( S(n,k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((5,1))</td>
<td>(\frac{1}{5!}(5)) = (1)</td>
<td>( S(5,1) = 1 )</td>
</tr>
<tr>
<td>((5,2))</td>
<td>(\frac{1}{2!1!}(5)) = (5)</td>
<td>( S(5,2) = 15 )</td>
</tr>
<tr>
<td>((3,2))</td>
<td>(\frac{1}{3!1!}(3)) = (10)</td>
<td>( S(5,3) = 25 )</td>
</tr>
<tr>
<td>((3,1,1))</td>
<td>(\frac{1}{2!1!1!}(3)) = (1/6 \cdot 120 = 20)</td>
<td>( S(5,4) = 10 )</td>
</tr>
<tr>
<td>((2,2,1))</td>
<td>(\frac{1}{2!1!1!}(2)) = (1/2 \cdot 120 = 60)</td>
<td>( S(5,5) = 1 )</td>
</tr>
<tr>
<td>((2,1,1,1))</td>
<td>(\frac{1}{1!3!1!}(1)) = (1/6 \cdot 120 = 20)</td>
<td>( S(5,6) = 1 )</td>
</tr>
</tbody>
</table>

Bell number \( B(5) = 52 \)

To enumerate partitions of \([n]\), first enumerate partitions of \(n\), then use Theorem 5.22 to say how many partitions of \([n]\) there are of each type. Check: the total number of partitions of \([n]\) is \( p(n) \); \([n]\) is \( B(n) \).
More examples with generating functions:

When rolling a pair of dice, any total from 2 to 12 can occur:

<table>
<thead>
<tr>
<th>Total roll</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1/36</td>
</tr>
<tr>
<td>3</td>
<td>2/36</td>
</tr>
<tr>
<td>4</td>
<td>3/36</td>
</tr>
<tr>
<td>5</td>
<td>4/36</td>
</tr>
<tr>
<td>6</td>
<td>5/36</td>
</tr>
<tr>
<td>7</td>
<td>6/36</td>
</tr>
<tr>
<td>8</td>
<td>5/36</td>
</tr>
<tr>
<td>9</td>
<td>4/36</td>
</tr>
<tr>
<td>10</td>
<td>3/36</td>
</tr>
<tr>
<td>11</td>
<td>2/36</td>
</tr>
<tr>
<td>12</td>
<td>1/36</td>
</tr>
</tbody>
</table>

\[ f(x) = x + \frac{x^2}{2} + \ldots + \frac{x^6}{6} \]

The generating function for the sequence 0, 1, 1, 1, 1, 1, 0, 0, 0, ... in which term \( n \) is the number of faces with label \( n \).

It is possible to construct a pair of non-standard cubic dice that achieve the same values (with the same probabilities) as the standard pair.

One die has faces: 1, 2, 3, 4, 5, 6
The second die has faces: 1, 2, 3, 4, 5, 6
\[
    f(x) = \frac{1}{(1-x)^3} = (x^0 + x^1 + x^2 + \cdots)(x^0 + x^2 + \cdots + x^6)
    = x^2 + 2x^3 + \cdots + 5x^6 + 6x^7 + 5x^8 + \cdots + 2x^{11} + x^{12}
\]

is the generating function for the sequence 0, 0, 1, 2, \ldots, 0, 0, 1, 2, \ldots, 0, 0, 1, 2, \ldots in which the \(n\)th term is the number of ways to roll \(n\) using a pair of dice.

Given \(n\) distinct beads, how many ways can we put them together to form a necklace?

<table>
<thead>
<tr>
<th>(n)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_n)</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>12</td>
<td>60</td>
<td>\ldots</td>
</tr>
</tbody>
</table>

\[
    a_n = \begin{cases}
        \frac{(n-1)!}{2}, & \text{if } n > 3; \\
        1, & \text{if } n \leq 3; \\
        0, & \text{if } n = 0.
    \end{cases}
\]

(no empty necklaces allowed)
\( C_n = \) number of ways to build a set of nonempty necklaces using \( n \) distinguishable beads

\( n = 1: \quad C_1 = 1 \)

\( n = 2: \quad \begin{align*} &C_2 = 2 \quad \text{or} \quad C_2 = 2 \end{align*} \)

\( n = 3: \quad \begin{align*} &C_3 = 5 \quad \text{or} \quad C_3 = 5 \quad \text{or} \quad C_3 = 5 \quad \text{or} \quad C_3 = 5 \quad \text{or} \quad C_3 = 5 \end{align*} \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_n )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>17</td>
<td>73</td>
<td>388</td>
<td>2461</td>
<td>18155</td>
<td>152581</td>
<td>...</td>
</tr>
</tbody>
</table>

**Exponential Generating Function**

\[
C(x) = \sum_{n=0}^{\infty} \frac{C_n}{n!} x^n = 1 + C_1 x + \frac{C_2}{2} x^2 + \frac{C_3}{3} x^3 + \cdots
\]

\[
A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = x + \frac{2}{2} x^2 + \sum_{n=3}^{\infty} \frac{(n-1)!}{2^n} x^n = x + \frac{x^2}{2} + \sum_{n=3}^{\infty} \frac{x^n}{2^n}
\]

\( a_n = \) no. of ways to construct a nonempty necklace using \( n \) beads

\[
a_n = \begin{cases} \frac{(n-1)!}{2} & \text{if } n \geq 3; \\ 1 & \text{if } n = 1,2; \\ 0 & \text{if } n = 0. \end{cases}
\]

The exponential generating function for \( a_n \) is

\[
A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = x + \frac{x^2}{2} + \sum_{n=3}^{\infty} \frac{x^n}{2^n}
\]
\[
A'(x) = 1 + x + \sum_{n=3}^{\infty} \frac{x^n}{2n} = 1 + x + \frac{1}{2} \sum_{n=3}^{\infty} x^n = 1 + x + \frac{1}{2} (x^2 + x^3 + x^4 + x^5 + \ldots)
\]

\[
= 1 + x + \frac{x^2}{2} (1 + x + x^2 + \ldots) = 1 + x + \frac{x^2}{2 (1-x)} = \frac{1}{2} + \frac{1}{2}x + \frac{1}{2} (1 + x + x^2 + x^3 + \ldots)
\]

\[
= \frac{1}{2} - \frac{1}{2}x + \frac{1}{2 (1-x)}
\]

\[
A(x) = \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{2} \ln(1-x) + \text{constant}
\]

\[
A(x) = \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{2} \ln(1-x).
\]

In general we are counting the number of ways to take \( n \) objects,

\[
\begin{bmatrix}
\begin{array}{c}
\text{1. partition them into any number of nonempty subsets;} \\
\text{2. put some structure on each subset;} \\
\text{3. put some other structure on the sets of the partition.}
\end{array}
\end{bmatrix}
\]

Structure could be:

- list in some order
- build a necklace
- designate one element of the set as special
- no additional structure

\( a_n = \text{no. of ways to put the first structure on a set of size } n \)

Assume \( a_0 = 0 \)

\( b_n = \ldots \text{ second} \ldots \)

\( c_n = \text{no. of ways to do the larger type of structure} \)

\( b_0 = 1 \)
\[ C(x) = B(A(x)) \] where \( A, B, C \) are the exponential generating functions for \( a_n, b_n, c_n \).

E.g., \( a_n \) is no. of ways to put the structure of a single necklace on a set of \( n \) beads.

\[ A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \frac{x}{2} + \frac{x^2}{4} - \frac{1}{2} \ln(1-x) \]

\[ b_n = 1 \text{ for all } n \] (no. of ways to put no. structure on a set of size \( n \)).

\[ B(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \]

\[ C(x) = B(A(x)) = e^{\frac{x}{2} + \frac{x^2}{4} - \frac{1}{2} \ln(1-x)} = \frac{e^{\frac{x}{2} + \frac{x^2}{4}}}{\sqrt{1-x}} = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n \]

Alternatively, let \( c_n \) be no. of ways to construct an ordered list of nonempty necklaces using \( n \) beads.

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_n )</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>13</td>
<td>57</td>
<td>271</td>
<td>1383</td>
<td>7875</td>
<td>46791</td>
</tr>
</tbody>
</table>

\[ a_n \text{ as before} \]

\[ b_n = n! \]

\[ B(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \]

\[ C(x) = B(A(x)) = \frac{1}{1-A(x)} = \frac{1}{1 - \frac{x}{2} - \frac{x^2}{4} + \frac{1}{2} \ln(1-x)} \]

\[ = 1 + x + \frac{3}{2} x^2 + \frac{13}{6} x^3 + \frac{77}{24} x^4 + \frac{193}{30} x^5 + ... \]
Simpler example: Let $c_n$ = no. of ways to simply partition a set of size $n$. (All structure is trivial: $a_n = 1$ ($n \geq 1$); $b_n = 1$, $n \geq 0$)

$$A(x) = \sum_{n=1}^{\infty} \frac{a_n}{n!} x^n = \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1$$

$$B(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$C(x) = B(A(x)) = e^{e^x - 1} = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n = \sum_{n=0}^{\infty} \frac{B(n)}{n!} x^n$$

$c_n$ = no. of ways to partition $n$ students into nonempty groups and then choose a leader in each group:

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_n$</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>10</td>
<td>?</td>
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</tbody>
</table>

$a_n = n$ is the number of ways to choose a leader in a group of $n$ students

$$A(x) = \sum_{n=1}^{\infty} \frac{a_n}{n!} x^n = \sum_{n=1}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} = x \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = x \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (n-1=k; \quad k=n+1)$$

$$= xe^x$$

$b_n = 1$ = no. of ways to put no structure on the collection of groups
\[ B(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x \]

\[ C(x) = B(A(x)) = e^{xe^x} = 1 + x + \frac{3}{2} x^2 + \frac{5}{6} x^3 + \frac{11}{24} x^4 + \cdots \]

\[
\begin{array}{cccccccc}
  n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\
  C_n & 1 & 1 & 3 & 10 & 41 & 196 & 1057 & \ldots \\
\end{array}
\]
Line up the four cubes in a row in such a way that all four colors appear on each of the four sides (front, back, top, bottom).

- What is the probability of solving the puzzle by pure chance?
- What is a systematic approach to solving the puzzle?

To answer the first question, I'll reveal that the solution is essentially unique, i.e., unique up to very trivial operations such as permuting the four cubes or rotating the entire row of cubes. With this understanding, what is the probability that the four cubes, dropped at random on a table and lined up in a row, will give a solution?
Graph: A graph has a set of vertices (any set of objects) of which every pair of vertices is either joined or not. Eg:

Vertices: four students: Al, Bob, Cal, Deb (A, B, C, D for short)
Relation: knowing each other. A \( \rightarrow \) B

\( \text{Diagram: } A \rightarrow B, B \leftrightarrow C, C \rightarrow D, D \rightarrow A \)

Relation is symmetric: A knows B if and only if B knows A

Relation is not symmetric:

- A knows C and D
- B knows C
- C knows A and B
- D knows A

(although the two pictures look different, the graphs are the same)

If one considers a relation that may not be symmetric:

- A knows B, C only
- B knows C
- C knows A, D
- D knows nobody

\( \text{Diagram: } A \rightarrow B, B \rightarrow C, C \rightarrow D, C \rightarrow A \)

This is a directed graph.
A graph may or may not include loops; edges from a vertex to itself:

Here is a graph with directed edges and loops representing a gift exchange: an edge from \( x \) to \( y \) means \( x \) buys a gift for \( y \).
The four instant insanity cubes give four undirected graphs with loops on vertices $G, B, W, R$ (the four colors), in which edges represent pairs of colors on opposite faces.

In any graph $Γ$ (Greek upper case gamma) the degree of a vertex $v$ is

$$\deg(v) = \text{number of edges coming out of } v.$$

To solve the instant insanity puzzle, we need to exclude one edge from each of the four graphs above, so that by putting the remaining eight edges together, get a graph of degree 4. Deleting the four edges marked in red above leads to a solution shown above, which is essentially unique.

Each graph has 3 edges and $\sum_{v} \deg(v) = 6$.
<table>
<thead>
<tr>
<th>Sunday</th>
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April 2016

Test 2

Read ch. 8, 9.
The Königsberg Bridge Problem:

How does one do a walking tour of this city that visits each bridge exactly once?

Impossible! as observed by Euler:

Consider a graph $\mathcal{G}$ whose vertices represent the four land masses and whose 7 edges represent the four bridges. This graph has odd degree $(3, 5, 3, 3)$ at every vertex.