Math 3700

Combinatorics

Book 3
Show me a graph on five vertices with degrees 2, 6, 4, 4, 2.

This graph has a closed Eulerian trail (Eulerian cycle).

An Eulerian trail is a tour that covers every edge in the graph; it is closed if it ends at the same vertex where it starts.
Show me a graph on six vertices with degrees 2, 2, 3, 4, 4, 5:

Any Eulerian trail in this graph must start at vertex 3 and end at vertex 5 or the other way around. (No closed Eulerian trail.)
Show me a graph on six vertices with degrees 2, 2, 3, 4, 4, 4:

There is no such graph.

**Theorem:** In any undirected graph (loops and multiple edges allowed), the sum of the vertex degrees equals twice the number of edges:

\[ \sum_{v \text{ vertex}} \deg(v) = 2 \times \text{no. of edges}. \]

**Proof:** Count the set of pairs \((x, e)\), \(x\) any vertex and \(e\) an edge containing \(x\). The number of such pairs is \(2\times\) the number of edges. On the other hand each vertex \(x\) lies in \(\deg(x)\) such pairs, so the number of pairs \((x, e)\) must be

\[ \sum_{x \text{ vertex}} \deg(x). \]

If \(m_1, m_2, \ldots, m_n\) are non-negative integers with \(m_1 + m_2 + \cdots + m_n\)

even, does there necessarily exist a graph with this degree sequence?

Yes, if we allow loops and multiple edges. This is a converse of the theorem above.
If a graph has every vertex of even degree, must it have a closed Eulerian trail (i.e. an Eulerian cycle)? No; e.g.

This graph is not connected, so it has no closed trail.

A graph is connected if for every pair of vertices \( x, y \) there is a path from \( x \) to \( y \).

**Theorem.** A graph has a closed Eulerian trail iff it is connected and every vertex has even degree.

**Proof.** It is clear that every graph having a closed Eulerian trail must be connected and every vertex has even degree. Conversely, suppose \( \Gamma \) is a connected graph (undirected, but we allow loops and multiple edges) in which every vertex has even degree. We need to show \( \Gamma \) has a closed Eulerian trail.
It is easy to find closed trails in $\Gamma$:

Since every closed trail uses only a finite number of edges, there must be a closed trail that uses as many edges as possible. We want to show that such a closed trail uses all the edges of $\Gamma$.

Reversing.
There is actually an efficient algorithm here (more than just an existence proof) for closed Eulerian trails (for simple connected graphs where every vertex has even degree).

A path (trail) in a graph $G$ that uses every vertex exactly once is a Hamiltonian path. Any such path that returns to the same vertex where it starts is a Hamiltonian cycle. 

This graph has a Hamiltonian cycle.

This graph has a Hamiltonian path but is there a Hamiltonian cycle? No; but to explain takes more work.
Given a simple graph $\Gamma$ (undirected) it is relatively easy to test whether $\Gamma$ is Eulerian (i.e., having a closed Eulerian trail) but in general, much harder to test whether $\Gamma$ is Hamiltonian (i.e., having a Hamiltonian cycle). More precisely:

Consider computational problems with yes/no answers (called decision problems).

1. Given a graph on $n$ vertices, does it have a Hamiltonian cycle?
2. “” “” “” “” “” “” “” “” “” closed Eulerian trail?
3. Given two $n$-digit numbers, does the first number divide the other?
4. Given an $n$-digit number, is it prime?
5. Given an $n$-digit number, is it even?

In each case the input is of variable size; what we are asking for is an algorithm to solve the problem in general. This algorithm (typically thought of as a computer program) requires more resources for larger input size $n$. (By resources we usually mean the computational time required; sometimes instead we mean the amount of memory required.)

(2), (3), (4), (5) all have efficient algorithms: the time required is no bigger than some polynomial in $n$. 

(27x377)
A decision problem has a polynomial time solution (in P) if there is an algorithm that answers the decision problem and it runs in time bounded by some polynomial in \( n \).

The problem of Hamiltonicity (testing whether a given graph has a Hamiltonian cycle) has no known polynomial time solution.

Testing for primality (i.e., given an \( n \)-digit number, is it prime) has a polynomial-time solution. Factoring does not.

Given two graphs, are they isomorphic?

---

**Examples:**

- Graph 1
- Graph 2
- Graph 3
- Graph 4
Show me two graphs with the same number of vertices of each degree, yet the graphs are not isomorphic?

\[ \Gamma_1 \neq \Gamma_3 \] (1 is joined to 3 in \( \Gamma_1 \) but not in \( \Gamma_3 \))

\[ \Gamma_1 \cong \Gamma_3 \]

Two graphs \( \Gamma, \Gamma' \) are isomorphic (denoted \( \Gamma \cong \Gamma' \)) if I can relabel the vertices of \( \Gamma \) to get \( \Gamma' \); more formally,

if there is a bijection from the vertices of \( \Gamma \) to the vertices of \( \Gamma' \) inducing a bijection from the edges of \( \Gamma \) to the edges of \( \Gamma' \).
Given two graphs, can you tell me if they are isomorphic?

Both graphs are regular of degree 3 (a graph is regular if every vertex has the same degree).

The Petersen graph has no cycles of length 4 but the second graph has cycles of length 4. Moreover the Petersen graph has no Hamiltonian cycle but the second graph does.

Any graph on 10 vertices which is regular of degree 3 and whose shortest cycles have length 5 is isomorphic to the Petersen graph.
Another important example: The complete graph on $n$ vertices is the ordinary graph having an edge between every pair of vertices, denoted $K_n$. eg.

$K_n$ has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges, the maximum number of edges in an ordinary graph on $n$ vertices (i.e. undirected graph with no loops or multiple edges).
Given an ordinary graph $\Gamma$ on $[n] = \{1, 2, \ldots, n\}$, the complement graph $\overline{\Gamma}$ on $[n]$ has the complementary set of edges: if $(i, j)$ is an edge in $\Gamma$ then it is a non-edge in $\overline{\Gamma}$ and conversely.

**Complete bipartite graph $K_{3, 3}$**
$K_{2,4}$

$K_{m,n}$ has $m+n$ vertices and $mn$ edges

$K_{2,2,8}$ (complete tripartite graph)

$K_{r,s,t}$ has $r+s+t$ vertices and $rs + rt + st$ edges

i.e., complement of
A graph is bipartite if its vertices can be partitioned as $X \cup Y$ such that all edges are between $X$ and $Y$ (not within $X$ or $Y$).

To test whether a given graph on $n$ vertices is bipartite can be done in polynomial time: it can be determined in time bounded by a polynomial in $n$.

To test whether two graphs are isomorphic has no known polynomial-time solution.
Conjectured hierarchy of computational difficulty (assuming NP ≠ P)
**Theorem** Let $\Gamma$ be a graph on $n$ vertices. If every vertex has degree $\geq \frac{n}{2}$ then $\Gamma$ has a Hamiltonian cycle.

**Proof** Suppose not, and let $\Gamma$ be a counterexample. We may assume that $\Gamma$ has as many edges as possible, i.e., add edges as long as possible without creating any Hamiltonian cycle. Note: $\Gamma \neq K_n$ (the complete graph) so there is a pair of vertices $x, y$ that are not joined in $\Gamma$. By our assumption, we cannot add an edge from $x$ to $y$ without creating a Hamiltonian cycle, there must be a Hamiltonian path from $x$ to $y$ in $\Gamma$.

![Diagram showing a Hamiltonian cycle](image)

**Strategy for finding a Hamiltonian cycle in $\Gamma$ (and hence a contradiction):** Find $i \in \{3, \ldots, n-1\}$ such that $x$ is joined to $z_i$ and $z_{i-1}$ is joined to $y$.

Let $A = \{ i \in \{3, n-1\} : x \sim z_i \}$ and $B = \{ i \in \{3, \ldots, n-1\} : z_{i-1} \sim y \}$.

$|A| \geq \frac{n}{2} - 1$ and $|B| \geq \frac{n}{2} - 1$ in a set of size $\{3, \ldots, n-1\} = n-3$. This violates $|A| + |B| \leq n-2$. 


Read Chapter 7: Inclusion - Exclusion

\[ |A \cup B| = |A| + |B| - |A \cap B| \]

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]
Given subsets $A_1, A_2, \ldots, A_n \subseteq S$,

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{k=1}^{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} (-1)^{k-1}|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$$

Example: There are $n$ participants in a gift exchange. Everyone puts their name on a slip of paper. All the names are put in a hat. Everyone then selects a slip of paper at random. There are $n!$ different outcomes (permutations of the $n$ individuals).

A desirable outcome would be a permutation $\pi$ of $1, 2, 3, \ldots, n$ (labels for the $n$ individuals) such that $\pi(i) \neq i$ for all $i \in [n]$. Such a permutation is a derangement.

Eg. $n=3$: There are six permutations of $[3] = \{1, 2, 3\}$ which we'll denote $123, 132, 213, 231, 312, 321$. Only two derangements: $231, 312$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Derangements of $[n]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>

We'll count derangements of $[n]$ as follows:
$A_i$ = set of all permutations $\pi$ of $[n]$ such that $\pi(i) = i$

$|A_i| = (n-1)!$

$|A_i \cap A_j| = (n-2)!$

$|A_i \cap A_{i_2} \cap \ldots \cap A_{i_k}| = (n-k)!$

This sum has $(\binom{n}{k})$ terms, each of which is $(n-k)!$

By Inclusion-Exclusion,

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|$$

No. of permutations that are not derangements =

$$\sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{(n-k)!} = \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!(n-k)!}$$

So the number of derangements of $[n]$ is

$$n! - n! \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = n! \left(1 - \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{24} + \frac{(-1)^n}{n!}\right)$$
For \( n = 1 \): 
\[
1!(1-1) = 0
\]

For \( n = 2 \): 
\[
2!(1 - 1 + \frac{1}{2}) = 1
\]

For \( n = 3 \): 
\[
3!(1 - 1 + \frac{1}{2} - \frac{1}{6}) = 6 \times \frac{1}{3} = 2.
\]

For \( n = 4 \): 
\[
4!(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}) = 24 \times 1 = 24
\]

If \( n \) individuals draw names out of a hat at random (each of the \( n! \) permutations occurring with the same probability \( \frac{1}{n!} \), i.e. the uniform distribution) the probability of this random permutation being a derangement is:

\[
\frac{\text{no. of derangements}}{\text{no. of outcomes}} = \frac{n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \ldots + \frac{(-1)^n}{n!}
\]

This value \( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1} = \frac{1}{e} \) \( (e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \text{ evaluated at } x = -1) \).
Another application of inclusion-exclusion: a formula for $S(n,k) = n!$ of partitions of $[n]$ into $k$ nonempty subsets. Recall: the number of functions from $[n]$ onto $[k]$ is $k! \cdot S(n,k)$. We can count this number using inclusion-exclusion as follows:

There are $k^n$ functions $[n] \to [k]$
There are $\frac{k^n}{(k-1)^n}$ functions $[n] \to [k]$ that never take the value $1$.
There are $\frac{k^n}{(k-2)^n}$ functions $[n] \to [k]$ that never take the value $1$ or $2$.

Define $A_i = \{\text{functions } [n] \to [k] \text{ that never take the value } i\}$, $1 \leq i \leq k$.

$|A_i| = (k-1)^n$
$|A_i \cap A_j| = (k-2)^n$ if $1 \leq i < j \leq k$
$|A_i \cap A_j \cap \ldots \cap A_r| = (k-r)^n$ if $1 \leq i < i_2 < \ldots < i_r \leq k$

$\{\text{Bad functions}\} = A_1 \cup A_2 \cup \ldots \cup A_k$

(i.e. not onto)

$|A_1 \cup A_2 \cup \ldots \cup A_k| = \sum_{r=1}^{k} (-1)^{k-r} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq k} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}|$

This sum has $(\binom{k}{r})$ terms, each of which is $(k-r)^n$

The number of "good functions" (onto)

$k! \cdot S(n,k) = k^n - \sum_{r=1}^{k} (-1)^{k-r} \binom{k}{r} (k-r)^n = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} (k-r)^n$
Eq. A computer screen has a large number of pixels, each composed of some combination of R, B, G, the three primary colors. If 30% of the pixels include red,
35% green,
20% blue,
10% both R and G,
8% R and B,
5% G and B, and
2% R, G and B,
how many pixels are black (no illumination)? 36%

30 + 35 + 20 - 10 - 8 - 5 + 2 = 64 so 64% of pixels are illuminated (i.e. by at least one of the three primary colors)

100% - 64% = 36% of pix are dark.
More general version of inclusion-exclusion:

Start with a set \([n] = \{1, 2, \ldots, n\}\) and two functions \(f, g\) defined on \(P([n])\) i.e. for each subset \(S \subseteq [n]\) we assign values \(f(S), g(S)\) subject to

\[
g(S) = \sum_{T \subseteq S} f(T) \quad \text{(the } f\text{-values determine the } g\text{-values)}.
\]

Then \(g\) also determines \(f\) via:

\[
f(S) = \sum_{T \subseteq S} (-1)^{\#S - \#T} g(T)
\]

E.g. \(n = 2; \ [2]\) has \(4\) subsets \(\emptyset, \{1\}, \{2\}, \{1, 2\}\).

\[
\begin{align*}
g(\emptyset) &= f(\emptyset) \\
g(\{1\}) &= f(\emptyset) + f(\{1\}) \\
g(\{2\}) &= f(\emptyset) + f(\{2\}) \\
g(\{1, 2\}) &= f(\emptyset) + f(\{1\}) + f(\{2\}) + f(\{1, 2\})
\end{align*}
\]

\[
\begin{bmatrix}
g(\emptyset) \\
g(\{1\}) \\
g(\{2\}) \\
g(\{1, 2\})
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix}
f(\emptyset) \\
f(\{1\}) \\
f(\{2\}) \\
f(\{1, 2\})
\end{bmatrix}
\]

\[
\begin{bmatrix}
g(\emptyset) \\
g(\{1\}) \\
g(\{2\}) \\
g(\{1, 2\})
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix}
g(\emptyset) \\
g(\{1\}) \\
g(\{2\}) \\
g(\{1, 2\})
\end{bmatrix}
\]
Application: Euler's formula for \( \phi(n) = \text{no. of integers } k \in \mathbb{N} \text{ such that } \gcd(k, n) = 1 \).

E.g., \( n = 12 \)

\[
\phi(12) = 12 - \left(\frac{12}{2}\right) - \left(\frac{12}{3}\right) + \left(\frac{12}{6}\right) = 4
\]

\[|\{1, 5, 7, 11\}| = 4\]

For the general case let \( p_1, \ldots, p_r \) are the distinct primes dividing \( n \):

\[
A_i = \{ k \in \mathbb{N} : p_i \mid k \}, \quad |A_i| = \frac{n}{p_i} \quad (\text{every } p_i \text{ number is a multiple of } p_i)
\]

\[
A_i \cap A_j = \{ k \in \mathbb{N} : k \text{ is divisible by both } p_i \text{ and } p_j \}, \quad |A_i \cap A_j| = \frac{n}{p_ip_j} \quad (\text{for } i \neq j)
\]

\[
|A_i \cap A_j \cap \ldots \cap A_r| = \frac{n}{p_1p_2\ldots p_r}
\]

"Bad" \( k \) i.e. \( \gcd(k, n) > 1 \):

\[
|A_1 \cup A_2 \cup \ldots \cup A_k| = \sum_{\emptyset \neq \mathcal{I} \subseteq [k]} (-1)^{|\mathcal{I}|} \sum_{I \in \mathcal{I}} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}| = \sum_{\emptyset \neq \mathcal{I} \subseteq [k]} (-1)^{|\mathcal{I}|} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq k} \frac{n}{p_{i_1}p_{i_2}\ldots p_{i_r}}
\]

"Good" \( k \) i.e. \( \gcd(k, n) = 1 \):

\[
\phi(n) = |\mathbb{N}| - |A_1 \cup A_2 \cup \ldots \cup A_k| = n - \sum_{\emptyset \neq \mathcal{I} \subseteq [k]} (-1)^{|\mathcal{I}|} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq k} \frac{n}{p_{i_1}p_{i_2}\ldots p_{i_r}} = n \sum_{r=0}^{k} \frac{1}{p_{i_1}p_{i_2}\ldots p_{i_r}}
\]

\[
= n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)
\]

UB: For \( k = 3 \)

\[
1 - \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p_3} + \frac{1}{p_1 p_2} + \frac{1}{p_1 p_3} + \frac{1}{p_2 p_3} - \frac{1}{p_1 p_2 p_3} = \frac{(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})(1 - \frac{1}{p_3})}{p_1 p_2 p_3}
\]
Eg. \( \phi(12) = 12 \cdot \frac{1}{2} \cdot \frac{2}{3} = 4 \)  
\( \phi(600) = 600 \cdot \frac{1}{2} \cdot \frac{2}{5} \cdot \frac{1}{2} = 240 \)

8 Partitions of 9 into distinct parts:

9, 8 + 1, 7 + 2, 6 + 3, 5 + 4, 6 + 2 + 1, 5 + 3 + 1, 4 + 3 + 2

\( q(9) = 8 \)

8 Partitions of 9 into odd parts:

9, 7 + 1 + 1, 5 + 3 + 1, 3 + 3 + 3, 5 + 1 + 1 + 1 + 1, 3 + 3 + 1 + 1, 3 + 1 + 1 + 1 + 1 + 1 + 1

\( p_0(9) = 8 \)

\( q(n) = \text{no. of partitions of } n \text{ into distinct parts} \)

\( p_0(n) = \text{no. of partitions of } n \text{ into odd parts} \)

Theorem. \( q(n) = p_0(n) \)

\[ Q(x) = \sum_{n=0}^{\infty} q(n) x^n = \prod_{k=1}^{\infty} (1 + x^k) = (1 + x)(1 + x^2)(1 + x^3)(1 + x^4)^n \ldots \]

\[ P_0(x) = \sum_{n=0}^{\infty} p_0(n) x^n = \prod_{k \text{ odd}} \frac{1}{1 - x^k} = \prod_{j=0}^{\infty} \frac{1}{1 - x^{2j+1}} (k = 2j + 1) \]

\[ = \frac{1}{1 - x} \cdot \frac{1}{1 - x^3} \cdot \frac{1}{1 - x^5} \cdot \frac{1}{1 - x^7} \ldots \]

\[ Q(x) = (1 + x)(1 + x^3)(1 + x^5)(1 + x^7)^n \ldots \times \frac{1 - x}{(1 - x)(1 - x^2)(1 - x^3)(1 - x^4)^n} \]

\[ = \frac{1}{(1 - x)(1 - x^3)(1 - x^5)(1 - x^7)^n} = P_0(x) \]

\( \square \)
\[ q_0(n) = \text{no. of partitions of } n \text{ into an odd number of distinct parts} \]
\[ q_e(n) = \text{no. of partitions of } n \text{ into an even number of distinct parts} \]
\[ q_e(n) + q_0(n) = q(n) = \text{no. of partitions of } n \text{ into distinct parts} \]
\[ q_e(9) = 4 = q_0(9), \quad q(9) = 8. \]
For most \( n \), \( q_e(n) = q_0(n) \).

**Theorem.** If \( n \) is not a pentagonal number, \( q_e(n) = q_0(n) \).

If \( n = \sum_j (j \in \mathbb{Z}) \) then \( q_e(n) = q_0(n) + (-1)^j \).

\[
\sum_{n=0}^{\infty} (q_e(n) + q_0(n)) x^n = Q(x) = \prod_{k=1}^{\infty} (1 + x^k) = (1 + x)(1 + x^2)(1 + x^3)(1 + x^4) \ldots \\
\sum_{n=0}^{\infty} \frac{q(n)}{\varphi(n)} x^n = \prod_{k=1}^{\infty} (1 - x^k) = (1 - x)(1 - x^2)(1 - x^3)(1 - x^4) \ldots \\
\sum_{n=0}^{\infty} (q_e(n) - q_0(n)) x^n \\
\]

**Eq.** Partitions of \( n = 4 \) into distinct parts 4, 3+1

\[
(1 + x)(1 + x^3)(1 + x^7)(1 + x^9) \ldots = 1 + x + x^2 + 2x^3 + 2x^4 + \ldots \\
(1 - x)(1 - x^2)(1 - x^3)(1 - x^4) \ldots = 1 - x - x^2 - x^3 - 2x^4 + 2x^5 + \theta x^6 + \ldots \\
\]
\[
\sum_{n=0}^{\infty} (q_n(x) - q_0(x)) x^n = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + \ldots
\]

\[
= \sum_{j=0}^{\infty} (-1)^j x^j \quad p_j = \frac{1}{2} j (3j-1) \quad (j^{th} \text{ pentagonal number})
\]

See handout for details.

Recurrence Formula for p(n):

\[
\sum_{n=0}^{\infty} p(n) x^n = \frac{1}{(1-x)(1-x^3)(1-x^4)(1-x^5) \ldots}
\]

\[
\Rightarrow 1 = \left( \sum_{n=0}^{\infty} p(n) x^n \right) \cdot (1-x)(1-x^2)(1-x^3)(1-x^4) \ldots
\]

\[
\underbrace{(p(0) + p(1)x + p(2)x^2 + p(3)x^3 + \ldots})
\]

\[
\underbrace{(1-x-x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + \ldots)}
\]

\[
= \sum_{j=0}^{\infty} (-1)^j x^j \quad p_j
\]

For \( n \geq 1, \quad 0 = p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - \ldots
\]

\[
\Rightarrow p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + \ldots
\]  

(Euler)
Let $G$ be an ordinary graph. Show that $G$ has two vertices of the same degree.

In other words, suppose we arrange a round robin tournament with $n$ competitors. At every point in time during the tournament, there are two competitors who have played the same number of matches.

Solution: See sample exam.

Eg. $n = 6$
Let $a_n$ be the number of connected graphs on $[n] = \{1, 2, \ldots, n\}$.

\[
\begin{array}{c|cccc}
 n & 1 & 2 & 3 & 4 & 5 \\
 a_n & 1 & 1 & 4 & 38 & \end{array}
\]

\[
\begin{align*}
\text{There are } 2^{\binom{n}{2}} \cdot \binom{n-1}{2} & \text{ graphs on } [n] \\
\text{Graphs on } [4] & \text{ with} \\
& \text{4 connected components} \\
& \text{3 connected components} \\
& \text{2 connected components} \\
& \text{1 connected component} \\
\end{align*}
\]

\[
\begin{align*}
& \text{1 such graph} \\
& \text{6 such graphs} \\
& \text{38 such graphs} \\
& \text{1 such } \square \text{ or } \square \text{ 15 such } \\
& \text{6 such } \vee \text{ or } \sqcup \text{ 4 such } \\
\end{align*}
\]

\[
\sum 2^{\binom{n}{2}} = 2^2 = 64 \text{ graphs in all}
\]

Total: 38 connected graphs
$A(x) = \sum_{n=1}^{\infty} \frac{a_n}{n!} x^n$  

exponential generating function for $a_n = \text{no. of connected graphs on } [n]$. 

$c_n = \text{no. of graphs on } n \text{ vertices} = 2^{\binom{n}{2}}$

$C(x) = \sum_{n=0}^{\infty} \frac{c_n}{n!} x^n = \sum_{n=0}^{\infty} \frac{2^{\binom{n}{2}}}{n!} x^n$ (I can't simplify this!) 

How many ways can we form a graph on $n$ vertices $[n]$? 

- First partition $[n]$ into $k$ nonempty subsets. Then: 
  - Make a connected graph on each of the $k$ subsets. 

The first step does not use any further structure i.e. $b_n = 1$ (trivial step)

$B(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$

$C(x) = B(A(x)) = e^{A(x)} \implies A(x) = \ln C(x) = \ln\left(\sum_{n=0}^{\infty} \frac{2^{\binom{n}{2}}}{n!} x^n\right)$

- See Maple worksheet