Counting Necklaces (Handout March, 2016)

We illustrate the use of exponential generating functions in a counting problem. More details are found in Chapter 8 of the textbook, although we will be covering only a few highlights of this topic.

How many ways can we string together \( n \) different beads to form a necklace? Start by stringing \( n \geq 2 \) beads onto a linear piece of string (i.e. a cord with two end points). There are \( n! \) choices of order for this sequence; but the reverse sequence gives exactly the same string of beads, so really there are

\[
\begin{cases}
  \frac{n!}{2} & \text{if } n \geq 2; \\
  1 & \text{if } n = 1
\end{cases}
\]

ways to string \( n \) beads onto a linear piece of string. (The case \( n = 1 \) is an exception, with only one way to string one bead, since reversing a string of one bead gives the same sequence.) Here we picture the \( \frac{4!}{2} = 12 \) ways to string 4 beads in a row:

![Image of 12 ways to string 4 beads in a row]

Now join the ends of the string together to form a loop. If \( n \geq 3 \), there are \( n \) different linear strings that could be used to form the same loop necklace (in other words, for each loop necklace there are \( n \) different points to break the string to form a linear string with \( n \)
beads). The cases \( n \leq 2 \) are again exceptional. So the number of loop necklaces that can be formed using \( n \) different beads is

\[
a_n = \begin{cases} 
\frac{n!}{2^n} = \frac{(n-1)!}{2} & \text{if } n \geq 3; \\
1, & \text{if } n = 1, 2.
\end{cases}
\]

By convention, we will take \( a_0 = 0 \) (with no beads, we cannot form a necklace at all). Here are the \( \frac{(4-1)!}{2} = 3 \) loop necklaces using 4 beads:

The exponential generating function for \( a_n \) is

\[
A(x) = \sum_{n \geq 1} \frac{a_n}{n!} x^n = x + \frac{1}{2} x^2 + \sum_{n \geq 3} \frac{(n-1)!}{2^n} x^n = x + \frac{1}{2} x^2 + \frac{1}{2} \left( \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots \right).
\]

To find a closed formula for \( A(x) \), consider the derivative

\[
A'(x) = 1 + x + \frac{1}{2} \left( x^2 + x^3 + x^4 + x^5 + \cdots \right) = \frac{1}{2} + \frac{1}{2} x + \frac{1}{2} \left( 1 + x + x^2 + x^3 + \cdots \right) = \frac{1}{2} + \frac{1}{2} x + \frac{1}{2} \left( \frac{1}{1-x} \right).
\]

Integrating, we obtain

\[
A(x) = \frac{x}{2} + \frac{x^2}{4} - \frac{1}{2} \ln(1-x).
\]

Now let \( c_n \) be the number of ways to form a collection of nonempty necklaces from \( n \) different beads. Call this number \( c_n \). After \( c_0 = 1 \) the next few terms are as shown:

\[
\begin{array}{ccc}
c_1 = 1 & c_2 = 2 & c_3 = 5 \\
\begin{array}{c}
1 \\
\end{array} & \begin{array}{c}
1 \ 2 \\
\end{array} & \begin{array}{c}
1 \ 2 \ 3 \\
1 \ 2 \ 3 \ 4 \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
c_4 = 17 \\
\begin{array}{c}
1 \ 2 \ 3 \ 4 \\
1 \ 2 \ 3 \ 4 \\
1 \ 2 \ 3 \ 4 \\
1 \ 2 \ 3 \ 4 \\
\end{array}
\end{array}
\]
No additional structure is required on the set of necklaces, so we take \( b_n = 1 \) and

\[
B(x) = \sum_{n \geq 0} \frac{b_n}{n!} x^n = \sum_{n \geq 0} \frac{x^n}{n!} = e^x.
\]

Finally, the exponential generating function of \( c_n \) is

\[
C(x) = B(A(x)) = \frac{e^{\frac{x^2}{2} + \frac{x^2}{4}}}{\sqrt{1 - x}} = 1 + x + x^2 + \frac{5}{6} x^3 + \frac{17}{24} x^4 + \frac{73}{120} x^5 + \frac{97}{180} x^6 + \ldots
\]

where we are able to determine as many terms as desired in this series expansion using the MAPLE session.

From this we are able to recover the value of \( c_n \) for small values of \( n \) by multiplying the coefficient of \( x^n \) in the series expansion, by \( n! \) as we have demonstrated in our MAPLE session, thereby obtaining

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 c_n & 1 & 1 & 2 & 5 & 17 & 73 & 388 & 2461 & 18155 & 152531 \\
\end{array}
\]

A second example, in which we incorporate more structure on the set of necklaces, is the following: Denote by \( c_n \) the number of ways to construct a set of loop necklaces from \( n \) different beads, and then arrange these necklaces in a row in order. We compute the first few terms in this new sequence by a slight modification of our first example. There
are now $b_n = n!$ ways to list $n$ objects in a row, and the exponential generating function for this is

$$B(x) = \sum_{n \geq 0} \frac{b_n}{n!} x^n = \sum_{n \geq 0} x^n = \frac{1}{1 - x}.$$ 

Using the same $A(x)$ as above, we now obtain

$$C(x) = B(A(x)) = \frac{1}{1 - \frac{x}{2} - \frac{x^2}{4} + \frac{1}{2} \ln(1 - x)} = 1 + x + \frac{3}{2} x^2 + \frac{13}{6} x^3 + \frac{77}{24} x^4 + \cdots.$$ 

Here is our MAPLE session:

As before, the first few values of $c_n$ are found to be

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_n$</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>13</td>
<td>77</td>
<td>572</td>
<td>5114</td>
<td>53406</td>
<td>637818</td>
</tr>
</tbody>
</table>

We verify the smallest values here using our previous catalog of necklaces:

$c_0 = 1$;
$c_1 = 1$;
$c_2 = 1 \times 2! + 1 \times 1! = 2 + 1 = 3$;
$c_3 = 1 \times 3! + 3 \times 2! + 1 \times 1! = 6 + 6 + 1 = 13$;
$c_4 = 1 \times 4! + 6 \times 3! + 7 \times 2! + 3 \times 1! = 24 + 36 + 14 + 3 = 77$. 

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Let us explain $c_3$: We have listed five ways to construct necklaces from 3 different beads,

- one way using three necklaces, which can be listed in $3! = 6$ ways;
- three ways using two necklaces, in each case with $2!$ ways to list the necklaces,
  for a total of $3 \times 2! = 6$ ways; and
- one way using a single necklace, which can be listed in only $1!$ way,
for a total of $c_3 = 6 + 6 + 1 = 13$ ways to construct necklaces from 3 different beads, then list them in some order.