Pentagonal Numbers (Handout May, 2016)

The familiar sequences of triangular square numbers have a natural geometric interpretation:

This pattern extends to an entire sequence of sequences: the polygonal numbers. For larger \( k \), the \( k \)-gonal numbers are less natural to motivate; but we find the sequence of pentagonal numbers worthy of special attention because of a surprising application to partition theory as we will soon discover:

Explicit formulas for triangular, square and pentagonal numbers, and the associated generating functions, are easily deduced:

\[
T_n = \frac{1}{2} n(n + 1) \quad S_n = n^2 \quad P_n = \frac{1}{2} n(3n - 1)
\]

\[
\sum_{n=0}^{\infty} T_n x^n = \frac{x}{(1-x)^3} \quad \sum_{n=0}^{\infty} S_n x^n = \frac{x(1+x)}{(1-x)^3} \quad \sum_{n=0}^{\infty} P_n x^n = \frac{x(1+2x)}{(1-x)^3}
\]

It is interesting to look at a table of values of \( T_n \), \( S_n \) and \( P_n \) over a range of integer values of \( n \), including some negative values of \( n \):

| \( n \) | \( -8 \) | \( -7 \) | \( -6 \) | \( -5 \) | \( -4 \) | \( -3 \) | \( -2 \) | \( -1 \) | \( 0 \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) | \( 7 \) | \( 8 \) |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \( T_n \) | 28    | 21    | 15    | 10    | 6     | 3     | 1     | 0     | 0     | 1     | 3     | 6     | 10    | 15    | 21    | 28    | 36    |
| \( S_n \) | 64    | 49    | 36    | 25    | 16    | 9     | 4     | 1     | 0     | 1     | 4     | 9     | 16    | 25    | 36    | 49    | 64    |
| \( P_n \) | 100   | 77    | 57    | 40    | 26    | 15    | 7     | 2     | 0     | 1     | 5     | 12    | 22    | 35    | 51    | 70    | 92    |

You should notice that the triangular and square numbers take the same values for \( n \leq 0 \) as for \( n \geq 0 \); this is explained by the relations

\[ T_{-n-1} = T_n, \quad S_{-n} = S_n \]
which are easily verified algebraically. In the case of pentagonal numbers, however, we get new values

\[ P_{-n} = \frac{1}{2}n(3n + 1) \]

which are not attained by the formula \( P_n = \frac{1}{2}n(3n - 1) \) for \( n > 0 \). We may therefore consider a pentagonal number to be any number of the form \( \frac{1}{2}n(3n \pm 1) \) for some \( n \geq 0 \); and it is now reasonable to rearrange the pentagonal numbers into a single sequence as 0, 1, 2, 5, 7, 12, 15, 22, . . . . Here we illustrate the first few pentagonal numbers graphically:

In the upper sequence \((n > 0)\) each pentagon of side \( n \) is composed of a square of side \( n \) and a triangle of side \( n - 1 \), giving

\[ P_n = S_n + T_{n-1} = n^2 + \frac{1}{2}n(n - 1) = \frac{1}{2}n(3n - 1); \]

while in the lower sequence, each pentagon is formed by a square and a triangle, both of side \( n \), yielding

\[ P_{-n} = S_n + T_n = n^2 + \frac{1}{2}n(n + 1) = \frac{1}{2}n(3n + 1). \]

Although these pentagons appear less symmetrical than those in the original geometric picture, the depiction here as as Ferrers diagrams relates more directly to our study of partitions.

**Partitions into Distinct Parts and Odd Parts**

Recall the partition function \( p(n) \), defined as the number of partitions of \( n \), i.e. the number of ways to write \( n \) as a sum of positive integers, where the order of the terms does not matter. We have seen that the ordinary generating function for \( p(n) \) is

\[
(\ast) \quad \sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k} = \prod_{k=1}^{\infty} \sum_{r_k=0}^{\infty} x^{r_k}k = \prod_{k=0}^{\infty} (1 + x^k + x^{2k} + x^{3k} + \cdots).
\]

Recall the explanation for this formula: after collecting terms on the right, the coefficient of \( x^n \) is the number of solutions of \( n = r_1 + 2r_2 + 3r_3 + \cdots \), i.e. the number of partitions of \( n \) in which there are exactly \( r_k \) parts of size \( k \); and since the limits on \( k \) and the \( r_k \)’s
ensure that we count every partition of \( n \) exactly once, the coefficient of \( x^n \) on the right side is \( p(n) \).

We now refine this counting problem by asking for \( q(n) \), the number of partitions of \( n \) into distinct parts; and \( p_o(n) \), the number of partitions of \( n \) into odd parts. For example, \( q(8) = p_o(8) = 6 \): the six partitions of 8 into distinct parts are

\[
8, \quad 5+2+1, \quad 4+3+1, \quad 7+1, \quad 6+2, \quad 5+3
\]

while the six partitions of 8 into odd parts are

\[
7+1, \quad 5+3, \quad 5+1+1+1, \quad 3+3+1+1, \quad 3+1+1+1+1, \quad 1+1+1+1+1+1+1+1.
\]

It is no coincidence that \( q(8) = p_o(8) \); in general we have

\[\textbf{Theorem 1.} \quad \text{The number of partitions of } n \text{ into distinct parts equals the number of partitions of } n \text{ into odd parts.}\]

\[\textbf{Proof.} \quad \text{The generating function for } q(n) \text{ and } p_o(n) \text{ are found by modifying (*) to restrict the type of partitions considered. For } q(n) \text{ we restrict to those partitions of } n \text{ having each term } k \text{ appear at most once, i.e. } r_k \in \{0, 1\}, \text{ which gives}
\]

\[
Q(x) = \sum_{n=0}^{\infty} q(n)x^n = \prod_{k=1}^{\infty} (1 + x^k) = (1 + x)(1 + x^2)(1 + x^3) \cdots
\]

\[
= 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + \cdots.
\]

For \( p_o(n) \) we restrict to those partitions of \( n \) having only odd terms \( k = 2j+1 \) appear, each appearing any number of times \( r_k \in \{0, 1, 2, 3, \ldots\} \), which gives

\[
P_o(x) = \sum_{n=0}^{\infty} p_o(n)x^n = \prod_{j=1}^{\infty} \frac{1}{1 - x^{2j+1}} = \frac{1}{(1 - x)(1 - x^3)(1 - x^5)(1 - x^9) \cdots}
\]

\[
= 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + \cdots.
\]

Using \texttt{Maple} we can verify that these two series agree to as many terms as desired, which certainly lends credibility to the statement we are trying to prove:
To prove Theorem 1, multiply numerator and denominator of the $P_o(x)$ expansion by $(1 - x^2)(1 - x^4)(1 - x^6) \cdots$ to obtain

$$P_o(x) = \frac{(1 - x^2)(1 - x^4)(1 - x^6)(1 - x^{10})(1 - x^{12}) \cdots}{(1 - x)(1 - x^2)(1 - x^3)(1 - x^5)(1 - x^6)(1 - x^7)(1 - x^8) \cdots}.$$  

After factoring each factor $1 - x^{2j} = (1 - x^j)(1 + x^j)$ in the numerator and cancelling factors with the denominator, we are left with

$$P_o(x) = (1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)(1 + x^6) \cdots = Q(x).$$

Comparing the coefficient of $x^n$ on each side gives $p_o(n) = q(n)$ as desired.

The preceding proof demonstrates the utility of generating functions; but we may be left to wonder why an algebraic proof should be needed to prove a strictly combinatorial fact. In fact a more combinatorial proof is possible. Such a proof would consist of an explicit bijection between the set of partitions of $n$ into distinct parts, and the set of partitions of $n$ into odd parts. This proof is a little less pretty (nothing as pretty as the use of conjugate partitions in giving a bijection between partitions of $n$ into $k$ parts, and partitions of $n$ into parts of maximum size $k$). Rather than giving all the details, we only sketch the proof and give $n = 8$ as an example: Given a partition of $n$ into distinct parts
as \( n = n_1 + n_2 + \cdots + n_k \), factor each \( n_i = 2^{c_i} m_i \) where \( c_i \geq 0 \) and \( m_i \) is the largest odd divisor of \( n_i \). Split \( n_i \) into \( 2^{c_i} \) parts of odd size \( m_i \) to obtain a partition of \( n \) into \( \sum_{i=1}^{k} 2^{c_i} \) parts. The \( m_i \)'s are not necessarily distinct (it is possible that \( m_i = m_j \) for some \( i \neq j \)); nevertheless we obtain a one-to-one correspondence between partitions of \( n \) into distinct parts, and partitions of \( n \) into odd parts, essentially because the binary representation of every positive integer is unique (i.e. there is only one way to write a given positive integer as a sum of distinct powers of 2). Here we illustrate this bijection in the case \( n = 8 \):

\[
\begin{align*}
8 &= 8 \cdot 1 \quad \leftrightarrow \quad 1+1+1+1+1+1+1+1 \\
7+1 &= 7+1 \\
6+2 &= 2 \cdot 3 + 2 \cdot 1 \quad \leftrightarrow \quad (3+3) + (1+1) \\
5+3 &= 5+3 \\
5+2+1 &= 5+2+1+1 \quad \leftrightarrow \quad (5) + (1+1) + (1) \\
4+3+1 &= 4 \cdot 1 + 3 + 1 \quad \leftrightarrow \quad (1+1+1+1) + (3) + (1+1)
\end{align*}
\]

Finally, for the promised connection to pentagonal numbers, we look at the pattern of even and odd coefficients in the generating function \( Q(x) = P_o(x) \). For this we simply reduce modulo 2:

What you should observe is that the exponents that appear in the latter sum are precisely the pentagonal numbers; i.e. \( q(n) = P_o(n) \) is odd if \( n \) is a pentagonal number, and even otherwise. The explanation for this observation is the following: Denote by \( q_e(n) \) and \( q_o(n) \)
the number of partitions of \( n \) into an even number of distinct parts, and an odd number of distinct parts, respectively, so that

\[ q_e(n) + q_o(n) = q(n). \]

**Theorem 2.** If \( n \) is not a pentagonal number, then \( q_e(n) = q_o(n) \) and so \( q(n) = 2q_o(n) \) which is even. If \( n \) is a pentagonal number, say \( n = P_j \), then \( q_e(n) = q_o(n) + (-1)^j \) and so \( q(n) = 2q_o(n) + (-1)^j \) which is odd.

From our enumeration of the six partitions of 8 into distinct parts, we have seen that \( q_e(8) = q_o(8) = 3 \) as predicted by Theorem 2 since 8 is not a pentagonal number. In the case \( n = 7 = P_{-2} \) we have \( q_e(7) = 3 \) partitions into an even number of distinct parts:

\[ 6+1, \quad 5+2, \quad 4+3; \]

and \( q_o(7) = 2 \) partitions into an odd number of distinct parts:

\[ 7, \quad 4+2+1, \]

as predicted by Theorem 2. In the case \( n = 12 = P_3 \), we have \( q_e(12) = 7 \) partitions into an even number of distinct parts:

\[ 11+1, \quad 10+2, \quad 9+3, \quad 8+4, \quad 7+5, \quad 6+3+2+1, \quad 5+4+2+1; \]

and \( q_o(12) = 8 \) partitions into an odd number of distinct parts:

\[ 12, \quad 9+2+1, \quad 8+3+1, \quad 7+4+1, \quad 7+3+2, \quad 6+5+1, \quad 6+4+2, \quad 5+4+3, \]

once again as predicted by Theorem 2.

The key to proving Theorem 2 is the following almost-bijective correspondence between partitions of \( n \) with an even number of parts, and partitions of \( n \) with an odd number of parts. Given a Ferrers diagram for a partition, denote by \( b \) the length of the bottom row (i.e. the size of the smallest part in the partition) and let \( r \) number of cells on the rightmost 45\(^\circ\) line. In the following example, we have \( b = 4 \) and \( r = 3 \):
If \( b \leq r \), move the bottom row to the rightmost 45° line; but if \( b > r \), move the rightmost 45° line down to the bottom. Here is one corresponding pair of partitions for \( n = 25 \):

and here is the complete correspondence for \( n = 8 \):

We obtain a well-defined bijection between partitions with an even number of distinct parts, and partitions with an odd number of distinct parts, except when \( n \) is a pentagonal number. If \( n = P_j \) where \( j > 0 \), then the correspondence fails just for the pentagonal Ferrers diagram with \( j \) rows having \( b = r = j \); whereas if \( n = P_{-j} \) where \( j > 0 \), then the correspondence fails just for the pentagonal Ferrers diagram with \( j \) rows having \( b = r + 1 \) and \( r = j \). Consider what happens in the cases \( n = P_4 = 22 \) and \( n = P_{-4} = 26 \) as shown:

When \( n \) is not a pentagonal number, no such pentagonal Ferrers diagram exists, and we obtain a well-defined bijection between partitions with an even number of distinct parts, and partitions with an odd number of distinct parts, giving \( q_e(n) = q_o(n) \). For a pentagonal number \( n = P_j \), there is just one left-over partition not covered by the bijection, and it has \( j \) parts, so \( q_e(n) = q_o(n) + (-1)^j \). This proves the theorem. \( \square \)

Just as

\[
Q(x) = \sum_{n=0}^{\infty} q(n)x^n = \sum_{n=0}^{\infty} (q_e(n) + q_o(n))x^n = (1 + x)(1 + x^2)(1 + x^3)(1 + x^4) \cdots ,
\]
we see that

\[(1 - x)(1 - x^2)(1 - x^3)(1 - x^4) \cdots = \sum_{n=0}^{\infty} (q_e(n) - q_o(n)) x^n\]
\[= \sum_{j=0}^{\infty} (-1)^j x^P_j\]
\[= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots\]

in which the only surviving terms are those whose exponents are pentagonal numbers! The reason is that positive terms \(x^n\) in the expansion of the latter product, correspond to partitions of \(n\) into an even number of distinct parts; whereas negative terms \(-x^n\) correspond to partitions of \(n\) into an odd number of distinct parts. Noting that the latter product is the reciprocal of

\[\sum_{n=0}^{\infty} p(n)x^n = \frac{1}{(1 - x)(1 - x^2)(1 - x^3)(1 - x^4) \cdots}.\]

we obtain the curious relation

\[(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots) \sum_{n=0}^{\infty} p(n)x^n = 1.\]

Comparing terms on both sides gives a recurrence formula for the partition function:

\[p(n) = p(n - 1) + p(n - 2) - p(n - 5) - p(n - 7) + p(n - 12) + p(n - 15) - \cdots\]

where we stop as soon as the argument becomes negative.