Sample Test 1
February, 2016

This sample test is intended to resemble the First Test (Monday, March 7, 2016 during class time) in approximate length, difficulty, and style, although clearly the content may differ. The actual content will consist of all material covered in class prior to the test, and all related handouts.

Instructions. The only aids allowed are a hand-held calculator and one ‘cheat sheet’, i.e. an 8.5” × 11” sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. Clarity is required for full credit! Time permitted: 50 minutes.

1. (20 points) Indicate how many of each of the following types of strings there are:
   (a) Bitstrings of length 10.
   (b) Bitstrings of length 10 with five 0’s and five 1’s.
   (c) Bitstrings of length 10 with no two consecutive 1’s.
   (d) Bitstrings of length 10 with no two consecutive 0’s and no two consecutive 1’s.

2. (20 points) Assume a PIN (personal identification number) is a string of four decimal digits 0,1,2,...,9. It is possible for a PIN to start with 0 (for example, 0737 is a valid PIN). Determine each of the following:
   (a) The number of possible PINs with arbitrary digits.
   (b) The number of possible PINs with distinct digits.
   (c) The number of possible PINs with distinct digits in strictly increasing order (i.e. of the form abcd where the digits satisfy a < b < c < d).
   (d) The number of possible PINs with digits in nondecreasing order (i.e. of the form abcd where the digits satisfy a ≤ b ≤ c ≤ d).

3. (10 points) Assuming the recurrence relation $F_n = F_{n-1} + F_{n-2}$ holds for all integer values of $n$, tabulate the values of $F_n$ for $-10 ≤ n ≤ 10$ given that $F_0 = 0$ and $F_1 = 1$. 
4. (20 points) For each item listed in the left column, find the closest match in the right column.

\begin{itemize}
\item[(i)] \( \frac{1}{1-x} \) \hspace{1cm} (A) \( 1 + x + x^2 + x^3 + x^4 + \ldots \)
\item[(ii)] \( \frac{1}{1+x} \) \hspace{1cm} (B) \( x + x^2 + 2x^3 + 3x^4 + 5x^5 + \ldots \)
\item[(iii)] \( \frac{x}{1-x-x^2} \) \hspace{1cm} (C) \( 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots \)
\item[(iv)] \( \frac{1}{1-x} \) \hspace{1cm} (D) \( 1 + x + 2x^2 + 3x^3 + 5x^4 + \ldots \)
\item[(v)] \( \frac{1}{(1-x)^2} \) \hspace{1cm} (E) \( x + x^2 + x^3 + x^4 + x^5 + \ldots \)
\item[(vi)] \( \frac{x}{1+x} \) \hspace{1cm} (F) \( 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots \)
\item[(vii)] \( (1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \ldots)^2 \) \hspace{1cm} (G) \( 1 + x + \frac{11}{12}x^2 + \frac{5}{6}x^3 + \ldots \)
\item[(viii)] \( \sqrt{1-x} \) \hspace{1cm} (H) \( 1 - x + x^2 - x^3 + \ldots \)
\item[(ix)] \( \frac{1}{\sqrt{1-x}} \) \hspace{1cm} (I) \( 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \ldots \)
\item[(x)] \( \frac{1}{(1-x)(1-x^2)(1-x^4)(1-x^4)^2} \) \hspace{1cm} (J) \( x - x^2 + x^3 - x^4 + \ldots \)
\end{itemize}
5. (30 points) Answer TRUE or FALSE to each of the following statements.

(a) The recurrence relation $F_{n+2} = F_{n+1} + F_n$ for Fibonacci numbers is proved by induction. 
   $(True/False)$

(b) For every positive integer $k$, the binomial coefficient $\binom{x}{k}$ is a polynomial in $x$ of degree $k$. 
   $(True/False)$

(c) In order for each UW student to have a unique W-number, the pigeonhole principle requires that W-numbers be more than 3 digits long. 
   $(True/False)$

(d) The Stirling numbers of the second kind satisfy the identity $S(n, n-k) = S(n, k)$. 
   $(True/False)$

(e) There are more than a billion different multisets whose elements come from the set $[10] = \{1, 2, 3, \ldots, 10\}$. 
   $(True/False)$

(f) Factorials satisfy the recurrence relation $n! = n(n-1)!$ for all $n \geq 1$. 
   $(True/False)$

(g) The Bell numbers satisfy the recurrence relation $B(n) = S(n, n) + B(n-1)$. 
   $(True/False)$

(h) In the approximation $n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$, the error is less than 0.01 whenever $n > 100$. 
   $(True/False)$

(i) The number of partitions of $n$ into odd parts equals the number of partitions of $n$ into an odd number of parts. 
   $(True/False)$

(j) The number of functions from $[n]$ to $[n]$ is $n!$. 
   $(True/False)$