Solutions to Sample Exam
May, 2016

1. (a) The values \( c_0 \) and \( c_1 \) are open to some debate and I would accept different answers here, assuming that in (b) and (c) you are consistent with these values. But if a Hamiltonian cycle is a connected subgraph in which every vertex has degree 2, then \( c_0 \) and \( c_1 \) must be as I have listed here.

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_n )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>12</td>
</tr>
</tbody>
</table>

(b) \( c_n = \begin{cases} 
1, & \text{if } n = 0; \\
0, & \text{if } n \in \{1, 2\}; \\
\frac{1}{2}(n-1)!, & \text{if } n \geq 3. 
\end{cases} \)

(c) Ordinary generating function: \( 1 + \frac{1}{2} \sum_{n=3}^\infty (n-1)!x^n \) which is already simplified as much as possible.

Exponential generating function: \( 1 + \frac{1}{2} \sum_{n=3}^\infty \frac{(n-1)!}{n!}x^n = 1 + \frac{1}{2} \sum_{n=3}^\infty \frac{x^n}{n} = 1 + \frac{1}{2}x + \frac{1}{3}x^2 - \frac{1}{2} \ln(1-x) \).

2. (a) \( 2^{\binom{5}{2}} = 2^{10} = 1024 \)
(b) 728 = 1024 - 296 since 296 is the number of disconnected graphs (first observe that there are 4 connected graphs on 3 vertices and 38 connected graphs on 4 vertices)
(c) 12 (they are all loop necklaces)
(d) 43 (12 isomorphic to \( \ graphite \), 20 isomorphic to \( \ circledast \), 10 isomorphic to \( \bigstar \) and 1 isomorphic to \( \bigcirc \))
(e) 120 (\( \binom{5}{3} = 10 \) isomorphic to \( \bigcirc \), 60 isomorphic to \( \bigtriangledown \), 30 isomorphic to \( \bigtriangledown \), and 20 isomorphic to \( \bigstar \))

Remark: With access to a computer, 2(b) is greatly simplified: Let \( a_n \) be the number of connected graphs on \([n]\), and \( A(x) = \sum_{n=1}^\infty \frac{a_n}{n!}x^n \) its exponential generating function. The total number of graphs on \([n]\) is \( 2^{\binom{n}{2}} \), whose exponential generating function is \( C(x) = \sum_{n=0}^\infty \frac{2^{n(n-1)/2}}{n!}x^n \). A graph on \([n]\) is formed by first partitioning the vertex set \([n]\) into an arbitrary number of nonempty subsets, then forming a connected graph on each subset. The only structure on the collection of subsets (i.e. connected components) is the trivial structure; and as we have seen before, there are \( b_n = 1 \) ways to put a trivial structure on
this collection and its exponential generating function is \( B(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n = e^x \). By the general theory, \( C(x) = B(A(x)) = e^{A(x)} \) and so Maple gives

\[
A(x) = \ln C(x) = x + \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{19}{12}x^4 + \frac{91}{15}x^5 + \cdots.
\]

In particular, \( a_5 = \left(\frac{91}{15}\right)5! = 728 \).

3. The number of residents using public transportation at least once was

\[
\sum_{J \neq \emptyset} (-1)^{|J|-1} 10^7 - |J| = \sum_{1 \leq j \leq 7} \left(\begin{array}{c} 7 \\ j \end{array}\right) (-1)^{j-1} 10^7 - j
\]

so the number of residents not using public transport at all last week was

\[
10^7 - \sum_{0 \leq j \leq 7} \left(\begin{array}{c} 7 \\ j \end{array}\right) (-1)^j 10^7 - j = 10^7 \sum_{0 \leq j \leq 7} \left(\begin{array}{c} 7 \\ j \end{array}\right) (-\frac{1}{10})^j
\]

\[
= 10^7 (1 - \frac{1}{10})^7 = 9^7 = 4,782,969.
\]

4. (a) \( g(3) = f(3) - f(1) \)
(b) \( g(4) = f(4) - f(2) \)
(c) \( g(6) = f(6) - f(3) - f(2) + f(1) \)
(d) \( g(12) = f(12) - f(6) - f(4) + f(2) \)
(e) \( g(100) = f(100) - f(50) - f(20) + f(10) \)

5. (a) \( a_n = \left(\begin{array}{c} n \\ 2 \end{array}\right) = \frac{n(n-1)}{2} \)
(b) \( \sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^n = \frac{x^2}{2} \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{x^2}{2} \frac{d^2}{dx^2} \sum_{n=0}^{\infty} x^n = \frac{x^2}{2} \frac{d^2}{dx^2} \frac{1}{1-x} \)
(c) \( \sum_{n=0}^{\infty} \frac{n(n-1)}{2n!} x^n = \frac{x^2}{2} \sum_{n=2}^{\infty} \frac{x^{n-2}}{(n-2)!} = \frac{x^2}{2} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{1}{2} x^2 e^x \)

6. \( S(n, k) \) is the number of ways to partition \([n]\) (or any \(n\)-set) into \(k\) nonempty subsets. This includes

(i) \( S(n-1, k-1) \) ways with \(\{1\}\) as one of the subsets and the remaining \(n-1\) numbers partitioned into \(k-1\) nonempty subsets; and

(ii) \( kS(n-1, k) \) ways to partition \(2, 3, \ldots, n\) into \(k\) subsets and then put 1 into one of the resulting sets.

So \( S(n, k) = S(n-1, k-1) + kS(n-1, k) \), the recurrence formula for \( S(n, k) \).

7. (a)F (b)T (c)T (d)T (e)F (f)F (g)T (i)F (j)T
Oops, sorry—I omitted (h). Here are some remarks and partial explanations for answers in #7:

(a) Most integer sequences do not have rational generating functions, for example the factorial sequence.

(b) The set of all edges of the graph is a Hamiltonian cycle.

(c) Color one vertex red and all its neighbors green. Color all neighbors of green vertices red. Continue coloring vertices alternately red and green until no new uncolored neighbors are found this way. If you ever encounter two adjacent vertices of the same color, the graph was not bipartite. If no uncolored vertices remain, the original graph was connected and bipartite; otherwise choose any uncolored vertex and color it red, its neighbors green, etc. repeating as above. This process terminates in time bounded by a polynomial in $n$, the number of vertices, and if it never finds two adjacent vertices of the same color, then the graph is bipartite.

(d) The fact that there is no injection $[n] \to [k]$ unless $n \leq k$, is one of the forms of the Pigeonhole Principle.

(e) The simple recurrence formula given is satisfied by the binomial coefficients. For the similar (but distinct) recurrence formula satisfied by $S(n, k)$, see #6.

(f) If a graph on 6 vertices has $e$ edges, its complement has $\binom{6}{2} - e = 15 - e$ edges, a number distinct from $e$ (it is even or odd according as $e$ itself is odd or even).

(g) Given a partition $\{A_1, A_2, \ldots, A_k\}$ of $[n]$, first express each $A_i$ as a list of integers in increasing order; then concatenate these lists in decreasing order of the first (i.e. smallest) elements of the individual $A_i$'s. This gives a single list of length $n$; and from this new list we can easily recover the original partition. So $B(n)$, the number of partitions of $[n]$, is at most $n!$, the number of lists that can be formed from $[n]$. In fact it is strictly less (i.e. $B(n) < n!$) since not every list arises from a partition. For example with $n = 5$, the list $2, 4, 5, 1, 3$ arises from the unique partition $\{\{2, 4, 5\}, \{1, 3\}\}$, but the list $2, 5, 4, 1, 3$ does not arise from any partition.

(i) $a_n = \binom{n+2-1}{n} = \frac{1}{2}(n+1)(n+2)$ which has only a polynomial growth rate

(j) For any graph on $n \geq 2$ vertices, every vertex has degree $d(v) \in \{0, 1, 2, \ldots, n-1\}$. If all $n$ vertices have distinct degrees, then some vertex $v$ has degree $n-1$, i.e. $v$ is joined to all other vertices. But then there can be no vertex of degree 0.