Solutions to HW1

1. a. $\alpha \beta^2 \alpha^{-1} \beta = (1\ 4)(3\ 5)$
   b. $\gamma = \alpha \beta^2 \alpha^3 \beta^2 \alpha$. There are many other correct expressions for $\gamma$ including $\alpha^3 \beta \alpha \beta \alpha^3$, $\alpha \beta \alpha^3 \beta \alpha \beta$, $\beta \alpha^3 \beta^2 \alpha^2 \alpha^3$, etc.

2. a. $\sigma \tau \sigma$: \[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \] is different from $\sigma \tau \sigma^{-1}$; in $\sigma \tau \sigma^{-1}$ the second strand lies behind the others, whereas in $\sigma^{-1} \tau \sigma$ the second strand lies in front of the others.
   b. $\tau \sigma \tau\sigma$: \[ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \]
   c. $\rho = \tau^{-1} \sigma = \sigma \tau \sigma^{-1} \tau^{-1}$

3. The elements in $GL_2(\mathbb{R})$ commuting with $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are precisely the matrices of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ where $a, b \in \mathbb{R}$ and $a \neq 0$. To verify this, suppose $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with $A$; then from $AX = XA$ we obtain $c = 0$ and $a = d \neq 0$, so $X$ has the claimed form. Conversely, every matrix of the form claimed clearly commutes with $A$.

4. There are exactly four elements of $S_4$ commuting with $(1\ 2)$, namely $()$, $(1\ 2)$, $(3\ 4)$ and $(1\ 2)(3\ 4)$.

5. There are exactly eight elements of $S_4$ commuting with $(1\ 2)(3\ 4)$, namely $(())$, $(1\ 2)$, $(3\ 4)$, $(1\ 2)(3\ 4)$, $(1\ 3\ 2\ 4)$, $(1\ 4\ 2\ 3)$, $(1\ 3)(2\ 4)$, $(1\ 4)(2\ 3)$.