1. (a) \(|G| = 8\).

(b) \(G\) has 1 element of order 1: \(1\)

5 elements of order 2: \((13), (24), (13)(24), (12)(34), (14)(23)\)

2 elements of order 4: \((1234), (1432)\).

(c) \(G\) has 1 subgroup of order 1:
\[\langle \langle 1 \rangle \rangle = \{ 1 \}\]

5 subgroups of order 2:
\[\langle \langle 13 \rangle \rangle, \langle \langle 24 \rangle \rangle, \langle \langle 13)(24) \rangle, \langle \langle 12)(34) \rangle, \langle \langle 14)(23) \rangle \]

3 subgroups of order 4:
\[\langle \langle 13, 24 \rangle \rangle, \langle \langle 12)(34), (13)(24) \rangle \text{ (Klein 4 groups)} \]
\[\langle \langle 13)(24) \rangle \text{ (cyclic of order 4)} \]

1 subgroup of order 8: \(G\) itself.

Note that \(G\) is dihedral of order 8: it is the symmetry group of the square \(3 \square 2\) represented as a permutation group on the four vertices, labelled as shown.

Remarks: To enumerate subgroups, start by first listing all cyclic subgroups, then subgroups generated by two elements. (In this case every subgroup is generated by two of its elements; otherwise we would proceed listing subgroups generated by three elements, four elements, etc.) Use Lagrange's theorem to reduce the work. For example if \(g, h \in G\) and one finds that \(\langle g, h \rangle\) contains at least five elements, then we conclude \(\langle g, h \rangle \neq G\) without any more work.
\#2. (a) \(|G| = 12\).

(b) \(G\) has 1 element of order 1: \(\langle 1 \rangle\)

3 elements of order 2: \((12)(34), (13)(24), (14)(23)\)

8 elements of order 3: \((123), (132), (124), (142), (134), (143), (234), (243)\)

(c) \(G\) has 1 subgroup of order 1: \(\langle 1 \rangle \rangle\)

3 subgroups of order 2: \(\langle (12)(34) \rangle, \langle (13)(24) \rangle, \langle (14)(23) \rangle\)

4 subgroups of order 3: \(\langle (123) \rangle, \langle (124) \rangle, \langle (134) \rangle, \langle (234) \rangle\)

1 subgroup of order 4: \(\langle (12)(34), (13)(24) \rangle\)

and 1 subgroup of order 12: \(G\) itself.

Remarks: The remarks after \#1 apply. Note that \(G\) has no subgroup of order 6. Indeed \(G = A_4\) of order \(\frac{9!}{2} = 12\) and this is known to be the smallest group for which the "converse" of Lagrange's Theorem fails.

One should use symmetry as much as possible to reduce the work involved; for example, given that \(G = \langle (123), (134) \rangle\), it follows by relabeling 1, 2, 3, 4 that any two distinct 3-cycles \(g, h\) (with \(h \neq g^{-1}\)) must also generate \(G\). Similarly, after verifying that \(\langle g, h \rangle = G\) for some pair of elements \(g, h\) of order 2 and 3 respectively, it easily follows (by relabeling) that the same is true for any pair of elements of order 2 and 3. Since two distinct elements
of order 2 must generate the Klein 4-subgroup. This actually makes light work of enumerating all subgroups of $G$. 