Test 1—Wednesday, March 8, 2017

Instructions. The only aids allowed are a hand-held calculator and one ‘cheat sheet’, i.e. an 8.5” × 11” sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. Clarity is required for full credit! Time permitted: 65 minutes (8:45–9:50 am). Total value of questions: 100 points (plus 10 bonus points).

1. (20 points) For each of the following objects, let $G$ be the full symmetry group and let $H \leq G$ be the subgroup consisting of rotational symmetries. In each case find $|G|$ and $|H|$; also indicate whether $G$ is abelian, and whether $H$ is abelian.

(a) A 2” × 3” × 8” brick

$|G| =$ \quad $|H| =$ \quad Is $G$ abelian? (yes/no) \quad Is $H$ abelian? (yes/no)

(b) A sphere

$|G| =$ \quad $|H| =$ \quad Is $G$ abelian? (yes/no) \quad Is $H$ abelian? (yes/no)

(c) A drill bit with two blades as shown:

$|G| =$ \quad $|H| =$ \quad Is $G$ abelian? (yes/no) \quad Is $H$ abelian? (yes/no)

(d) A circle

$|G| =$ \quad $|H| =$ \quad Is $G$ abelian? (yes/no) \quad Is $H$ abelian? (yes/no)
2. \textbf{(20 points)} Let $G = GL_2(\mathbb{R})$, the multiplicative group of all $2 \times 2$ matrices with real entries.

(a) Give an explicit example of an element of order 4 in $G$.

(b) Find a Klein 4-subgroup of $G$.

(c) Give an explicit example of an element of infinite order in $G$.

(d) Find a non-identity element in $G$ which commutes with every element of $G$. 
3. (20 points) Consider the symmetric group $S_6$.
   
   (a) What is the order of $S_6$?

   (b) How many elements of order 5 does $S_6$ have? Explain.

   (c) How many elements of order 6 does $S_6$ have? Explain.
4. (20 points) Let $G$ be the multiplicative group consisting of all matrices of the form \[
\begin{bmatrix}
a & b \\
0 & 1
\end{bmatrix}
\] where $a, b \in \mathbb{R}$ and $a \neq 0$.

(a) The group $G$ has a subgroup $H$ isomorphic to the additive group of real numbers, $\mathbb{R}$. Find such a subgroup $H \leq G$ and an isomorphism $\phi : \mathbb{R} \to H$.

(b) The group $G$ has a subgroup $K$ isomorphic to the multiplicative group of nonzero real numbers, $\mathbb{R}^\times$. Find such a subgroup $K \leq G$ and an isomorphism $\psi : \mathbb{R}^\times \to K$. 
5. (30 points) Answer TRUE or FALSE to each of the following statements. In (a)–(c), assume that \( x, y \) are elements in a multiplicative group \( G \).

(a) The subgroups \( \langle x, y \rangle \) and \( \langle xy, y^{-1} \rangle \) coincide, i.e. \( \langle xy, y^{-1} \rangle = \langle x, y \rangle \).

(b) The order of \( xy \) is necessarily the least common multiple of the orders of \( x \) and \( y \).

(c) If \( G = \langle x, y \rangle \) where \( xy = yx \), then \( G \) is necessarily abelian.

(d) The symmetry group of square contains the four vertices and the four sides of the square.

(e) Every subgroup of a cyclic group is cyclic.

(f) Every group of order at most 5 is abelian.

(g) The braid group \( B_n \) has a subgroup isomorphic to the symmetric group \( S_n \).

(h) The group \( S_7 \) has a subgroup isomorphic to \( S_6 \).

(i) The set of all nonzero integers forms a multiplicative group.

(j) The additive group of real numbers is cyclic, generated by the element 1.