Homework Assignment HW5  Due Friday, March 13, 2015

A1 Prove that there are infinitely primes congruent to 5 modulo 6. Your proof should mimic closely our proof that there are infinitely many primes congruent to 3 modulo 4 (as presented on the class; also given in the textbook on p.86).

This exercise is not a good time to strive for originality, since our proof that there are infinitely many primes congruent to 3 modulo 4 was already close to optimal: just enough details, but no extra. The intended value of this exercise is much the same as in learning any other language through mimicry, with only slight changes in content as required by the different arithmetic progression 5,11,17,23,29,... in place of 3,7,11,15,19,23,...

A2 Can the argument used in A1 be easily modified to show that there are infinitely many primes congruent to 1 modulo 6, i.e. in the arithmetic progression 1,7,13,19,25,...? Explain.

A3 Our proof that there are infinitely many primes of the form 4k + 1, can be modified to show that there are infinitely many primes of the form 6k + 1, using the Dirichlet series

\[ L_0(x) = 1 + \frac{1}{5^x} + \frac{1}{7^x} + \frac{1}{11^x} + \frac{1}{13^x} + \frac{1}{17^x} + \frac{1}{19^x} + \frac{1}{23^x} + \frac{1}{25^x} + \cdots \]

and

\[ L_1(x) = 1 - \frac{1}{5^x} + \frac{1}{7^x} - \frac{1}{11^x} + \frac{1}{13^x} - \frac{1}{17^x} + \frac{1}{19^x} - \frac{1}{23^x} + \frac{1}{25^x} - \cdots \]

(a) Write down the Euler factorization of each of these series, showing explicitly the first few Euler factors.

(b) For which real values of \( x \) does each of the series \( L_0(x) \), \( L_1(x) \) converge?

(c) What is the behavior of each of these series as \( x \to 1^+ \)?

Proofs are not required in A3, as these results follow from Calculus. All that is expected is a clear statement of fact, similar to what was given in our proof in the case of the primes modulo 4.

A4 Find all solutions of \( \phi(n) \in \{2010,2011,\ldots,2020\} \). (Here \( \phi \) is Euler’s totient function.) Justify your answers.