1. (a) \( \phi(2015) = (5-1)(13-1)(31-1) = 1440 \)

(b) \( \gcd(k, 2015) = 5 \) iff \( k = 5\ell, \ 1 \leq \ell \leq 403, \gcd(\ell, 403) = 1. \) The number of such \( \ell \) is \( \phi(403) = (13-1)(31-1) = 360. \)

(c) \( \sigma(2015) = (5+1)(13+1)(31+1) = 2688 \)

2. (a) \( \frac{\pi(10^{100})}{10^{100}} \approx \frac{1}{\ln(10^{100})} = \frac{1}{100 \ln 10} \approx 0.0043, \) i.e. 0.43%

(b) \( \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \approx 0.61, \) i.e. 61%

(c) \( \frac{\pi(10^{100})}{40 \times 10^{100}} \approx \frac{10}{4 \ln(10^{100})} = \frac{1}{40 \ln 10} \approx 0.011, \) i.e. 1.1%

3. 

<table>
<thead>
<tr>
<th>71</th>
<th>31</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>(-2)</td>
</tr>
<tr>
<td>(-3)</td>
<td>7</td>
</tr>
<tr>
<td>7</td>
<td>(-16)</td>
</tr>
</tbody>
</table>

so \( \gcd(31, 71) = 1 \) and all solutions of the congruence are given by \( x \equiv -16 \equiv 55 \mod 71. \)

4. (a) \( M \) is divisible by 3 only when \( p = 2, \ M = 3; \) otherwise \( p \) is odd so \( M = 2^p - 1 \equiv 2 - 1 \equiv 1 \mod 3. \)

(b) \( M \) is never divisible by 5. If \( p = 2 \) then \( M = 3; \) otherwise \( p \) is odd so \( 2^p \equiv 2 \) or \( 3 \) \mod 5, so \( M \equiv 1 \) or \( 2 \) \mod 5.

(c) \( M \) is never divisible by \( p \) itself since by Fermat’s Little Theorem, \( M = 2^p - 1 \equiv 2 - 1 \equiv 1 \mod p. \)

5. The primitive Pythagorean triple \( (p, \frac{1}{2}(p^2-1), \frac{1}{2}(p^2+1)) \) works. To find this solution, let \( p = m^2 - n^2 = (m+n)(m-n) \) so \( m-n = 1 \) and \( m+n = p. \) Solve to obtain \( m = \frac{1}{2}(p+1) \) and \( n = \frac{1}{2}(p-1), \) then substitute into \( (m^2-n^2, 2mn, m^2+n^2). \)
Although you are not required to explain your answers to True/False questions, the following remarks may help to understand the solution key:

1. It is an open problem (known as Goldbach’s Conjecture) to show that $n$ is a sum of two primes.
2. We presented Euclid’s proof in class.
3. It is not known whether or not there are infinitely many pairs of twin primes.
4. It is not known whether or not there are infinitely many primes of the form $N^2 + 1$.
5. If $p$ is an odd prime then $2^p + 1 \equiv 2 + 1 \equiv 0 \mod 3$.
6. Since $\gcd(123456789, 1000000000) = 1$, there are infinitely many primes $p$ satisfying $p \equiv 123456789 \mod 1000000000$ by Dirichlet’s Theorem.
7. The function $\chi$ is not multiplicative since $\chi(4) \neq \chi(2)\chi(2)$.
8. Use the Alternating Series Test (Leibniz’ Test).
9. State-of-the-art technology allows us to factor arbitrary integers of at most a couple hundred decimal digits.
10. This was covered in class, and on the relevant handout.