Straightedge-and-Compass Constructions

Starting with a designated unit length, the following lengths may be constructed using straightedge and compass: all rational lengths, and those lengths constructible from previously constructed lengths using the operations of addition, subtraction, multiplication, division, and taking square roots. The numbers $\pi$ and $\sqrt{2}$ are not of this form, so no line segment of either such length is constructible using straightedge and compass. For similar reasons, it is not possible to trisect an arbitrary given angle. In fact, it is not possible to construct a $20^\circ$ angle (i.e. to trisect a $60^\circ$ angle; or equivalently, to construct a regular nonagon i.e. 9-gon) since $\cos(20^\circ)$ is a root of $8x^3 - 6x + 1 = 0$; to construct the roots of this polynomial equation requires the extraction of cube roots and so is not possible to implement using straightedge and compass. A course in algebraic number theory or field theory is, however, required in order to understand the details of this argument.

It is possible to construct a regular $n$-gon using a straightedge and compass, if and only if $\cos\left(\frac{2\pi}{n}\right)$ is one of the constructible lengths as described above. For example a regular pentagon is constructible since $\cos(72^\circ) = \cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5} - 1}{4}$ which is a constructible length. The point of this homework assignment is to verify this identity.

Other values of $n$ for which a regular $n$-gon can be constructed using a straightedge and compass are

a) $n = 3, 4, 5, 17, 257, 65537$;

b) products of distinct values in the list (a) above, e.g. $3 \times 5 = 15$ and $3 \times 17 \times 257 = 13107$ but not $3 \times 3 = 9$ or $5 \times 5 = 25$;

c) values of $n$ obtained from those in (a) or (b) by multiplying by a power of 2, e.g. $2^3 \times 3 \times 5 = 120$.

In (a) we have listed all known Fermat primes; these are primes which are 1 more than a power of 2. We have listed all known values of $n$ for which such a construction is possible; and unless any more Fermat primes are discovered (which seems unlikely), this list is complete.

**HW 12**

Verify that $\cos(72^\circ) = \cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5} - 1}{4}$. You may use any familiar trigonometric identities such as

$\sin^2 \theta + \cos^2 \theta = 1$

$\sin(2\theta) = 2 \sin(\theta) \cos(\theta)$

$\cos(2\theta) = 2 \cos^2(\theta) - 1$

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$