Discussion of HW1

We use fairly standard notation which denotes by $AB$ the line joining points $A$ and $B$; the line segment joining $A$ and $B$ is denoted $\overline{AB}$. In order to define line segments, we require a notion of betweenness, which exists in the Euclidean setting but not in the projective plane setting. Following the wording of the homework question, which asks for suitable extensions of line segments, we consider lines rather than line segments. We write $\ell \parallel m$ to denote that lines $\ell$ and $m$ are parallel; and we write $\ell \nparallel m$ to denote that lines $\ell$ and $m$ are not parallel.

A more precise statement of Pascals Theorem is the following: Let $\gamma$ be a nondegenerate conic, and let $A, B, C, D, E, F$ be distinct points of $\gamma$. Then the three points

$$P = AB \cap DE, \quad Q = BC \cap EF \quad \text{and} \quad R = CD \cap FA$$

are collinear. My wordy statement of Pascals Theorem, which was intended to facilitate your conceptualization of the theorem, instead caused a fair amount of confusion—probably there is a lesson there. I referred to $ABCDEF$ as a hexagon inscribed in $\gamma$, in which case each of the points $P, Q, R$ is the intersection of a pair of opposite sides of the hexagon. However, some participants reasonably understood a hexagon as necessarily having six distinct vertices. Whatever you mean by the term hexagon, this should not affect the question of whether or not the conclusion holds (namely that $P, Q, R$ are collinear). I also avoided giving formal names and definitions of the points $P, Q$ and $R$, simply saying that these three new points (to distinguish them from the six points $A, B, C, D, E, F$ that we started with) lie on a single line, i.e. they are collinear. Unfortunately some participants interpreted new to mean that these new points $P, Q, R$ are distinct from each other, or distinct from the original points $A, B, C, D, E, F$. This meaning was unintended, and it caused some students to lose track of the original question, namely whether $P, Q, R$ must be collinear. Sorry about that! In the future I'll try to stick to more technical language or notation. Here are some of the key observations I was looking for, as we seek a generalization of Pascals Theorem:

With the original placement of distinct points $A, B, C, D, E, F$ on the conic $\gamma$, as shown in class, the conclusion always holds, in both the Euclidean plane and its extension (the real projective plane). With a different placement of six distinct points $A, B, C, D, E, F$ on the conic $\gamma$, the conclusion of Pascals Theorem holds every time in the real projective plane (where points at infinity are included). In the Euclidean plane, the conclusion fails (in the strictest sense) whenever a pair of opposite sides is parallel (i.e. $AB \parallel DE$ or $BC \parallel EF$ or
However, the following wording suggests an interpretation of the conclusion where validity is restored: In the Euclidean case,

(i) If \( AB \parallel DE \) and \( BC \parallel EF \) and \( CD \parallel FA \), then the three points

\[
P = AB \cap DE, \quad Q = BC \cap EF \quad \text{and} \quad R = CD \cap FA
\]

are collinear.

(ii) If \( AB \parallel DE \) and \( BC \parallel EF \), then \( CD \parallel FA \).

If \( BC \parallel EF \) and \( CD \parallel FA \), then \( AB \parallel DE \).

If \( CD \parallel FA \) and \( AB \parallel DE \), then \( BC \parallel EF \).

(iii) If \( AB \parallel DE \) and \( BC \not\parallel EF \), then \( CD \not\parallel FA \), so we may consider the points

\[
Q = BC \cap EF \quad \text{and} \quad R = CD \cap FA.
\]

In this case the line \( QR \) is parallel to the lines \( AB \) and \( DE \).
A similar conclusion holds if $BC \parallel EF$ and $CD \parallel FA$; or in several other such similar cases.

The important point is that by speaking about points at infinity, we can take a theorem of Euclidean geometry like the one above (which requires a vast proliferation of statements to cover all the different situations that can arise, depending on which pairs of lines are parallel) and replace them with one succinct theorem. Although from the Euclidean viewpoint such ‘points at infinity’ do not exist, in the real projective plane they actually exist.

A great deal of simplification occurs in the projective setting:

- As above, we avoid having to consider troublesome special cases that may arise due to parallel lines.
- We see ellipses, hyperbolas and parabolas as merely different ways of looking at a single type of nondegenerate conic.
- Every theorem yields another theorem (its dual) for free, with no need for another proof.
- Only in the projective setting does the exact form of Bezout’s theorem hold (which counts the number of intersection points of two curves). Without counting points at infinity, it is possible that we are missing some points of intersection.