Discussion of HW7

Everyone managed to find coordinates for points and lines of the projective plane of order 3. Here is Sirena’s illustration, for example:

We observe that this is a projective plane; but why it works may have been a mystery. In particular, why should \((x, y, z)\) and \((\lambda x, \lambda y, \lambda z)\) represent the same point? And why should we be representing points by triples \((x, y, z)\) rather than pairs \((x, y)\)? We explain this as follows. Since the general construction works for any field in place of the integers mod 3, I will explain using the field \(\mathbb{R}\) of real numbers.

The Real Projective Plane

Denote by \(\mathbb{R}^3\) the set of all triples \((x, y, z)\) of real numbers. Think of this as Euclidean 3-space. Define a **Point** to be a line of \(\mathbb{R}^3\) through the origin; and define a **Line** to be a plane of \(\mathbb{R}^3\) through
the origin. Note that any two Points lie on exactly one Line (since in \( \mathbb{R}^3 \), any two lines through the origin determine a unique plane through the origin). Also any two Lines meet in exactly one Point (since in \( \mathbb{R}^3 \), any two planes through the origin intersect in a unique line through the origin). This shows that our new geometry of Points and Lines satisfies the first two axioms (P1) and (P2) of projective plane geometry; and the third axiom (P3) is easily checked (i.e., our geometry is not degenerate). This geometry of Points and Lines is the real projective plane. At least it’s one description of the real projective plane; and in a moment we will fiddle with this description to obtain another one that may be more convenient in some situations.

But first let’s remark, using this construction, that the real projective plane is very homogeneous: any two Points behave the same way. Likewise, any two Lines behave the same way. This was not obvious in our previous construction of the real projective plane, in which we started with the Euclidean plane and added new points at infinity, and a new line at infinity. Such a construction gives the impression that the ‘points at infinity’ are somehow different from the ordinary points of the Euclidean plane. Our new construction of the real projective plane (if it is actually the same thing… and in fact, it is) shows that viewing some points as being ‘at infinity’ is just a distorted viewpoint; viewed more naturally, any two points look the same.

**Coordinates for Points and Lines in Classical Planes**

Now to describe our construction in another way, remember that a Point is a line through the origin in \( \mathbb{R}^3 \), and so consists of all scalar multiples \((\lambda x, \lambda y, \lambda z)\) of a fixed nonzero vector \((x, y, z)\).

Similarly, a Line is a plane through the origin in \( \mathbb{R}^3 \), which we denote by a column vector \(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\) which represents the set of all Points \((x, y, z)\) satisfying \(ax + by + cz = 0\). Here again, every scalar multiple of \(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\) yields a linear equation with the same solutions, and so must represent the same Line.

All of this generalizes to an arbitrary field \(F\) (think of the field of order 3 from HW7, the instance at hand)... and I will go back to writing ‘point’ and ‘line’ in lower case. The classical construction of a projective plane coordinatized by \(F\) has points given by nonzero vectors \((x, y, z)\), where nonzero scalar multiples represent the same point; and lines represented by nonzero column vectors \(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\), again where nonzero scalar multiples represent the same line. Incidence happens whenever the equation \(ax + by + cz = 0\) holds.

**The Real Projective Plane Constructed from a Sphere**

Let’s return to the real projective plane, as described above. Each Point is a line through the origin in \( \mathbb{R}^3 \), and it is natural to use a unit vector to represent such a line. However there are two choices of unit vector \( \pm \mathbf{u} \) for each line. Denote by \(S\) the unit sphere in \( \mathbb{R}^3 \); then each Point of
the real projective plane corresponds to a pair of antipodal points $\pm \vec{u}$ on $S$. In other words, points of the unit sphere $S$ represent Points of the real projective plane; but we must identify antipodal points. Similarly each Line is a plane through the origin, and this is represented by a great circle of $S$ (not just any old circle on $S$, but rather the intersection of $S$ with a plane through the origin). Let’s check once again that our axioms are satisfied: Any two points $A, B \in S$ determine a unique great circle of $S$ (formed by intersection $S$ with the plane containing $A, B$ and the origin). (The uniqueness fails if $A$ and $B$ are antipodal; but in that case, they represent the same Point!) So (P1) holds. And any two great circles of $S$ intersect in a pair of antipodal points; and these constitute a single Point! So (P2) holds. And (P3) is easily checked.

**The Real Projective Plane Constructed from a Hemisphere**

Starting with the latter construction of the real projective plane from a sphere $S$, we next observe that the points of the lower (‘southern’) hemisphere are redundant, since they represent the same Points as the upper (‘northern’) hemisphere under the identification of each point with its antipode. So let’s discard the redundant points and retain only the upper hemisphere. We must remember, however, that there remains some redundancy: opposite points on the equator still need to be identified. To recap: the real projective plane has Points represented by points of a hemisphere, but with antipodal points of the boundary (‘equator’) identified.

**The Real Projective Plane Constructed from a Hemisphere**

Another simplification is possible: Projecting a hemisphere down to a disk, we may (at least topologically) view the real projective plane as a disk, with antipodal points on its boundary identified. This is the first view of the real projective plane we encountered, given in Day 3 of our course.