MATH 5640 — SOLUTIONS TO HW5

#1. (a) From $(x-a)^2+y^2=r^2$ we get $2(x-a)dx+2y\,dy=0$ so $dy=\frac{a-x}{y}dx$. Now

$$ds^2=\frac{dx^2+dy^2}{y^2}=\left[1+\left(\frac{a-x}{y}\right)^2\right]\frac{dx^2}{y^2}=\frac{r^2}{y^4}dx^2,$$

i.e. $ds^2=\dfrac{r^2\,dx^2}{[r^2-(x-a)^2]^2}$. We make an arbitrary convention to parameterize the geodesic from left to right, so that $\frac{ds}{dx}>0$; then

$$ds=\frac{r}{r^2-(x-a)^2}dx.$$

Substitute $x-a=r\tanh u$, so that $dx=r\,\mathrm{sech}^2u\,du$ and

$$ds=\frac{r^2\,\mathrm{sech}^2u\,du}{r^2-r^2\tanh^2u}=\frac{r^2\,\mathrm{sech}^2u}{r^2\,\mathrm{sech}^2u}du=du.$$

Thus $s-s_0=u$ and we again make an arbitrary choice of parameter s, to assure that $s_0=0$. (This means we measure s from the point $(x,y)=(0,r)$.) Thus $s=u$, $x-a=r\tanh u$, $y=\sqrt{r^2-r^2\tanh^2u}=r\,\mathrm{sech}\,u$.

(b) The points $(1,1)$ and $(-1,1)$ lie on the geodesic $x^2+y^2=2$ parameterized by

$$(x,y)=(\sqrt{2}\tanh s,\ \sqrt{2}\,\mathrm{sech}\,s).$$

Solving $\mathrm{sech}\,s=\frac{1}{\sqrt{2}}=\frac{2}{e^s+e^{-s}}$, we have $e^{2s}-2\sqrt{2}e^s+1=0$ so $e^s=\sqrt{2}\pm1$ and $s=\pm\ln(1+\sqrt{2})$. The required distance is $2\ln(1+\sqrt{2})\approx1.76275$.

<u>Check</u>: The fractional linear transformation $z \mapsto \frac{\sqrt{2}+z}{\sqrt{2}-z}$ maps $\sqrt{2} \mapsto \infty$, $-\sqrt{2} \mapsto 0$ and hence maps the upper semicircle $|z|=\sqrt{2}$, $\operatorname{Im} z>0$ to the upper imaginary axis $\operatorname{Re} z=0$, $\operatorname{Im} z>0$. It is an isometry mapping

$$(1,1) = 1+i \mapsto (1+\sqrt{2})i \, ; \qquad (-1,1) = -1+i \mapsto (\sqrt{2}-1)i.$$

So the distance from $(1,1)$ to $(-1,1)$ equals the distance between $(\sqrt{2}+1)i$ and $(\sqrt{2}-1)i$. As shown on p. 6 of the handout on Hyperbolic Planes, this distance is

$$\ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right| = \ln(\sqrt{2}+1)^2 = 2\ln(\sqrt{2}+1).$$

This agrees with the answer found above.

Our approach depends on finding a fractional linear transformation mapping the indicated geodesic $|z|=\sqrt{2}$ to a vertical line. But this is easily accomplished by mapping one of the endpoints $(\pm\sqrt{2}, 0)$ to ∞. So any fractional linear transformation with denominator $z \pm \sqrt{2}$ will do.

#7. Note that M is a torus ($T^2 = S^1 \times S^1$). It is possible to use just three local charts; but the coordinates are easier with four local charts, as follows:

$$f_1(s,t) = (\cos s, \sin s, \cos t, \sin t), \quad 0 < s < 2\pi, \; 0 < t < 2\pi;$$

$$f_2(s,t) = (\cos s, \sin s, \cos t, \sin t), \quad -\pi < s < \pi, \; 0 < t < 2\pi;$$

$$f_3(s,t) = (\cos s, \sin s, \cos t, \sin t), \quad 0 < s < 2\pi, \; -\pi < t < \pi;$$

$$f_4(s,t) = (\cos s, \sin s, \cos t, \sin t), \quad -\pi < s < \pi, \; -\pi < t < \pi.$$

The corresponding domains are open rectangles, chosen so that each f_i is injective. The four images have union equal to the entire torus M. Each $f_i^{-1} \circ f_j$ is the identity on its domain; and this is <u>very</u> smooth (C^∞; in fact real analytic) as required.

#12. This exercise lays a foundation for later use of "geodesic coordinates" (see p. 277). Write

$$(x^0, x^1, \cdots, x^n) = (t, x^1, \cdots, x^n).$$

The new metric is

$$\begin{bmatrix} g_{00} & g_{01} & \cdots & g_{0n} \\ g_{10} & g_{11} & \cdots & g_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n0} & g_{n1} & \cdots & g_{nn} \end{bmatrix} = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & f(t)^2 g^*_{11} & \cdots & f(t)^2 g^*_{1n} \\ \vdots & \vdots & & \vdots \\ 0 & f(t)^2 g^*_{n1} & \cdots & f(t)^2 g^*_{nn} \end{array}\right]$$

We require $f(t) \neq 0$ (so that g is positive definite).

We use the formula

$$\Gamma_{ijk} = \frac{1}{2}\left[\frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ik} - \frac{\partial}{\partial x^k} g_{ij}\right]$$

to compute

$$\Gamma_{ijk} = f(t)^2 \, \Gamma^*_{ijk} \quad \text{when } i, j, k \geq 1;$$

$$\Gamma_{0jk} = f(t) f'(t) \, g^*_{jk} \quad \text{when } j, k \geq 1;$$

$$= \Gamma_{j0k} = -\Gamma_{jk0}$$

and $\Gamma_{i00} = \Gamma_{0j0} = \Gamma_{00k} = 0$ for all i, j, k.

$$\text{Also } g^{ij} = \begin{cases} 1, & \text{if } i = j = 0; \\ 0, & \text{if } i = 0 < j \text{ or } i > 0 = j; \\ f(t)^{-2} (g^*)^{ij}, & \text{if } i, j \geq 1. \end{cases}$$

Now

$$\Gamma_{ij}^{\ k} = \Gamma_{ijr}\, g^{rk} = \underbrace{\cancel{f(t)}^{2}\, \Gamma^{*}_{ijr}\, \cancel{f(t)}^{-2} (g^{*})^{rk}}_{\text{sum over } r=1,2,\cdots,n} = (\Gamma_{ij}^{\ k})^{*}$$

for $i,j,k \geq 1$ (so most ~~of~~ the Christoffel symbols are unchanged). Also

$$\Gamma_{ij}^{\ 0} = \Gamma_{ij0} = -f(t)\, f'(t)\, g^{*}_{ij} \qquad \text{for } i,j \geq 1;$$

$$\Gamma_{i0}^{\ k} = \Gamma_{i0r}\, g^{rk} = f(t)\, f'(t)\, g^{*}_{ir}\, f(t)^{-2} (g^{*})^{rk} = \frac{f'(t)}{f(t)}\, \delta_i^{\ k} = \Gamma_{0i}^{\ k}$$

for $i, k \geq 1$.

The geodesic equations from (p. 143) become

$$\begin{cases} \ddot{x}^{k} + (\Gamma^{*})_{ij}^{\ k}\, \dot{x}^{i} \dot{x}^{j} + 2\, \dfrac{f'(t)}{f(t)}\, \dot{x}^{k} \dot{t} = 0, & k \geq 1; \\ \ddot{t} - f(t)\, f'(t)\, g^{*}_{ij}\, \dot{x}^{i} \dot{x}^{j} = 0. & \end{cases}$$

Geodesics in M^{*} with $t =$ constant give $\dot{t} = \ddot{t} = 0$, so $\ddot{x}^{k} + (\Gamma^{*})_{ij}^{\ k}\, \dot{x}^{i} \dot{x}^{j} = 0$ is satisfied. But then the second equation is not satisfied <u>unless $f(t)$ is constant</u>, since g^{*} is positive definite and $g^{*}_{ij}\, \dot{x}^{i} \dot{x}^{j} > 0$.

On the other hand if t is not constant, then we may conveniently use t to parameterize our curves, so that $\dot{t}=1$ and $\ddot{t}=0$. The second geodesic equation will be satisfied if $g^*_{ij}\dot{x}^i\dot{x}^j=0$, which (since g^* is positive definite) implies that $(\dot{x}^1,\cdots,\dot{x}^n)=(0,0,\cdots,0)$, i.e. $x^1,\cdots,x^n$ are constant. In this case the first geodesic equations $\ddot{x}^k +$ etc. $=0$ $(k=1,2,\cdots,n)$ are clearly also satisfied. This verifies the conclusion that "t-lines are geodesics" as desired.

#20. The metric tensor is

$$(g_{ij}) = \begin{bmatrix} -h & 0 \\ 0 & \frac{1}{h} \end{bmatrix}, \quad (g^{ij}) = \begin{bmatrix} -\frac{1}{h} & 0 \\ 0 & h \end{bmatrix}, \quad h = 1-\frac{r_0}{r}$$

The map $\phi : E \to E$, $(t,r) \mapsto (\pm t+b, r)$ (b constant) has Jacobian

$$D\phi = \begin{bmatrix} \frac{\partial \phi_1}{\partial t} & \frac{\partial \phi_1}{\partial r} \\ \frac{\partial \phi_2}{\partial t} & \frac{\partial \phi_2}{\partial r} \end{bmatrix} = \begin{bmatrix} \pm 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So

$$(D\phi)^T (g_{ij}) (D\phi) = \begin{bmatrix} \pm 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -h & 0 \\ 0 & \frac{1}{h} \end{bmatrix} \begin{bmatrix} \pm 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -h & 0 \\ 0 & \frac{1}{h} \end{bmatrix} = (g_{ij})$$

i.e. ϕ is an isometry.

We have $\frac{\partial}{\partial r} g_{tt} = -\frac{\partial}{\partial r} h = \frac{r_0}{r^2}$

$$\frac{\partial}{\partial r} g_{rr} = \frac{\partial}{\partial r}\left(\frac{1}{h}\right) = -\frac{r_0}{r^2 h^2} = -\frac{r_0}{(r-r_0)^2}$$

and all other first order partial derivatives of the metric tensor components vanish, giving

$$\Gamma_{rtt} = \frac{1}{2}\frac{\partial}{\partial r} g_{tt} = \frac{r_0}{2r^2} = \Gamma_{trt} = -\Gamma_{ttr}$$

$$\Gamma_{rrr} = \frac{1}{2}\frac{\partial}{\partial r} g_{rr} = -\frac{r_0}{2r^2 h^2}$$

and all other Γ_{ijk} vanish. The only nonvanishing Christoffel symbols are

$$\Gamma^r_{rr} = \Gamma_{rrr} g^{rr} = -\frac{r_0}{2r^2 h}$$

$$\Gamma^r_{tt} = \Gamma_{ttr} g^{rr} = -\frac{r_0 h}{2r^2}$$

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \Gamma_{rtt} g^{tt} = -\frac{r_0}{2r^2 h}$$

The geodesic equations are
$$\begin{cases} \ddot{t} - \frac{r_0}{r^2 h}\dot{r}\dot{t} = 0 & \text{(i)} \\ \ddot{r} - \frac{r_0}{2r^2 h}\dot{r}^2 = 0 & \text{(ii)} \end{cases}$$

The t-lines have the form $r =$ constant, so $\dot{r} = \ddot{r} = 0$. They may be parameterized by t, so that $\dot{t} = 1$, $\ddot{t} = 0$. Clearly these values satisfy (i) and (ii), i.e. t-lines are geodesics.

We also check from $h = 1 - \frac{r_0}{r}$, $h' = -\frac{r_0}{r^2} r'$ that

$$(ht')' = ht'' + h't'$$

$$= \frac{r_0}{r^2} r't' - \frac{r_0}{r^2} r't' \qquad \text{(using (i))}$$

$$= 0$$

so ht' is constant along geodesics.