Embedding Finite Partial Linear Spaces in Finite Projective Planes

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A partial linear space (PLS) is a pair $\Gamma = (\mathcal{P}, \mathcal{L})$ consisting of a set $\mathcal{P}$ (points) and a collection $\mathcal{L}$ of distinguished subsets of $\mathcal{P}$ (called lines) such that

(i) each line contains at least two points, and

(ii) any two distinct lines meet in at most one point.

A point-line pair $(P, \ell)$ in $\Gamma$ is called a flag or an antiflag according as $P \in \ell$ or $P \notin \ell$. 
Let \( \Gamma = (\mathcal{P}, \mathcal{L}) \) and \( \tilde{\Gamma} = (\tilde{\mathcal{P}}, \tilde{\mathcal{L}}) \) be two partial linear spaces.

An embedding \( \alpha : \Gamma \to \tilde{\Gamma} \) is a pair of injections

\[
\alpha_1 : \mathcal{P} \to \tilde{\mathcal{P}}, \quad \alpha_2 : \mathcal{L} \to \tilde{\mathcal{L}}
\]

such that for all \( P \in \mathcal{P}, \ell \in \mathcal{L}, \)

\[
P \in \ell \Rightarrow \alpha_1(P) \in \alpha_2(\ell).
\]

Such an embedding is strong if

\[
P \in \ell \iff \alpha_1(P) \in \alpha_2(\ell).
\]
Every incidence system may be identified with its *point-line incidence graph*.

Thus a PLS corresponds to a bipartite graph of girth $\geq 6$ (i.e. no 4-cycles).

An embedded PLS corresponds to a subgraph.

A strongly embedded PLS corresponds to an induced subgraph.
Theorem. Every PLS is strongly embeddable in an infinite projective plane.

Proof. Use free closure.

Question. Can every finite PLS be embedded in a finite projective plane?

Proposition. The following two statements are logically equivalent.

(i) Every finite PLS is embeddable in a finite projective plane.

(ii) Every finite PLS is strongly embeddable in a finite projective plane.
Our Survey Says...

Francis Buekenhout polled participants at a recent conference if they believed that every finite linear space is embeddable in a finite projective plane.

22 voted YES

2 voted NO

19 abstained
A partial linear space $\Gamma$ is *saturated* if no edges can be added to its incidence graph without creating a 4-cycle.

**Lemma.** *Every finite PLS $\Gamma$ is strongly embeddable in a saturated finite PLS.*

**Proof.** Join every antiflag $(P, \ell)$ in $\Gamma$ by a path of length three (using two new vertices per antiflag). Then add further edges (as necessary) until the graph is saturated. 

\[
\begin{array}{c}
\text{A} \quad \text{u} \quad \text{B} \\
\end{array}
\]

\[
\begin{array}{c}
\text{C} \\
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\]

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\begin{array}{c}
\text{C} \\
\end{array}
\]
**Proposition.** The following two statements are logically equivalent.

(i) Every finite PLS is embeddable in a finite projective plane.

(ii) Every finite PLS is strongly embeddable in a finite projective plane.

**Proof.** (ii)⇒(i) is trivial.

(i)⇒(ii): Let Γ be a finite PLS. First strongly embed Γ → Γ₁ where Γ₁ is a saturated finite PLS. Then embed Γ₁ → Π where Π is a finite projective plane. The composite

Γ → Γ₁ → Π

is necessarily a strong embedding of Γ.
PLS: the class of all finite partial linear spaces;

PROJ: the class of all point-line incidence systems formed by subsets of finite projective planes;

TRANS: the class of all point-line incidence systems formed by subsets of finite projective translation planes;

NET: the class of all point-line incidence systems formed by subsets of finite translation nets (arising from partial spreads).
A finite PLS which *cannot* be embedded in any *Desarguesian* plane:

Another:
Theorem. For every $d \geq 1$, there exists a finite PLS $\Gamma$ which is not embeddable in any André plane of dimension $\leq d$ over its kernel.
André Planes

Let $E \supset F$ be an extension of finite fields, $|E| = q^d$, $|F| = q$.

The automorphism group

$$Aut(E/F) = \{1, \sigma, \sigma^2, \ldots, \sigma^{d-1}\}$$

where $\sigma : E \to E$, $x \mapsto x^q$.

The norm map $N : E \to F$, $x \mapsto x^{1+q+q^2+\ldots+q^{d-1}}$

Chose any map $\phi : F^\times \to Aut(E/F)$ s.t. $\phi(1) = 1$. There are $d^{q-2}$ choices for $\phi$.

Each $\phi$ gives an André plane with

$q^{2d}$ points $(x, y) \in E \times E$;

$q^d(q^d + 1)$ lines $x = a$ (for $a \in E$);

$$y = mx^{\phi(N(m))} + b$$ (for $m, b \in E$).
Essence of Proof that this is a plane

Try to find a line through two given points \((x_1, y_1), (x_2, y_2)\) with \(x_1 \neq x_2\)

\[
y = mx + b
\]

\[
y_2 - y_1 = m(x_2 - x_1)
\]

\[
\Rightarrow N(y_2 - y_1) = N(m)N(x_2 - x_1)
\]

\[
\Rightarrow N(m) = \frac{N(y_2 - y_1)}{N(x_2 - x_1)} \text{ is determined}
\]

\[
\Rightarrow m = \frac{y_2 - y_1}{(x_2 - x_1) \phi(N(m))} \text{ is determined}
\]

\[
\Rightarrow b \text{ is determined}
\]
**Theorem.** For every $d \geq 1$, there exists a finite PLS $\Gamma$ which is not embeddable in any André plane of dimension $\leq d$ over its kernel.

**Lemma.** Given a finite PLS $\Gamma$ and an integer $d \geq 1$, there exists a finite PLS $\hat{\Gamma}$ such that for every $d$-colouring of the lines, there is an embedded copy of $\Gamma$ with all lines having the same colour.

**Proof.** This follows from a result of Nešetřil and Rödl.

**Proof of Theorem.** Let $\Gamma$ be a finite PLS not embeddable in any Desarguesian plane. Let $\hat{\Gamma}$ be as in the Lemma. Suppose $\hat{\Gamma}$ embeds in an André plane $\mathcal{A}$ as above.

Then $\Gamma$ is embedded in $\mathcal{A}$ in such a way that for some fixed $\alpha \in Aut(E/F)$, all lines of $\Gamma \subset \mathcal{A}$ have the form $y = mx^\alpha + b$ for some $m, b \in E$.

Apply $(x, y) \mapsto (x^{\alpha^{-1}}, y)$ to give an embedding of $\Gamma$ in a Desarguesian net, a contradiction.
Open Questions

Does there exist a finite PLS which is not embeddable in any translation net of dimension 2 over its kernel?

Does there exist a finite PLS which is not embeddable in any André plane?

Is there a good criterion for embeddability of a finite PLS in a finite Desarguesian plane?
\( y = M_1 x + b_1 \)  
\( y = M_2 x + b_2 \)  
\( y = M_3 x + b_3 \)  
\( y = M_4 x + b_4 \)  
\( y = M_5 x + b_5 \)

\[
M_1 x_1 - M_1 x_2 + M_2 x_2 - M_2 x_3 + M_3 x_3 - M_3 x_4 + M_4 x_4 - M_4 x_5 + M_5 x_5 - M_5 x_1 = 0
\]
Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a finite PLS. A reasonable approach to embedding $\Gamma$ in a finite translation net follows:

$C_0(\Gamma) = \text{the } \mathbb{Z}\text{-module freely generated by } \mathcal{P} \cup \mathcal{L}$

$C_1(\Gamma) = \text{the } \mathbb{Z}\text{-module freely generated by the flags of } \Gamma$

Consider the complex

$$0 \xrightarrow{\delta} C_1(\Gamma) \xrightarrow{\delta} C_0(\Gamma) \xrightarrow{\delta} 0$$

where $\delta(P, \ell) = P - \ell$ for each flag $(P, \ell)$. The Euler characteristic of this complex is

$$\dim H_0(\Gamma) - \dim H_1(\Gamma) = \dim C_0(\Gamma) - \dim C_1(\Gamma)$$

Here

$\dim H_0(\Gamma) = \text{number of connected components};$

$\dim H_1(\Gamma) = \text{dimension of the circuit space};$

$\dim C_0(\Gamma) = \text{total number of points and lines};$

$\dim C_1(\Gamma) = \text{number of flags of } \Gamma.$
Given $\Gamma$, we want to find a finite vector space $V$ such that $\Gamma$ embeds in an affine translation net in $V \oplus V$. We seek functions

$$f : \mathcal{P} \to V \quad \text{and} \quad g : \mathcal{L} \to E := \text{End}(V)$$

such that

(i) $f$ is injective;

(ii) for all $\ell \neq \ell'$ in $\mathcal{L}$, we have $g(\ell) - g(\ell') \in E^\times = GL(V)$; and

(iii) $H_1(\Gamma) = Z_1(\Gamma)$ is contained in the kernel of the linear map

$$f \times g : C_1(\Gamma) \to V$$

defined by $(P, \ell) \mapsto g(\ell)f(P)$ for every flag $(P, \ell)$ of $\Gamma$.

This yields a translation net (i.e. partial spread) of $V \oplus V$ with components

$$S_\ell := \{(v, g(\ell)v) : v \in V\} \quad \text{for} \quad \ell \in \mathcal{L}$$

in which $\Gamma$ is embedded.