Transfer Matrix Method

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Reference: Transfer Matrix Method

References: Dimensions of Codes


M. Bardoe and P. Sin, ‘The permutation modules for $GL(n+1,q)$ acting on $P^n(q)$ and $F_q^{n+1}$’, to appear in JLMS.

http://www.math.ufl.edu/~sin/preprints/hamada.dvi

G.E. Moorhouse, ‘Dimensions of Codes from Finite Projective Spaces’ (as html and as Maple worksheet)

http://math.uwyo.edu/~moorhous/src/hamada.html
http://math.uwyo.edu/~moorhous/src/hamada.mws
Problem 1

Let \( S_k \) be the set of ‘words’ of length \( k \) consisting of ‘a’s and ‘b’s, with no two consecutive ‘b’s. Determine \( F_k = |S_k| \).

\[
\begin{align*}
F_0 &= 1 & F_1 &= 2 & F_2 &= 3 & F_3 &= 5 & F_4 &= 8 \\
'' & & 'a' & & 'aa' & & 'aaa' & & 'aaaa' \\
'b' & & 'ab' & & 'aab' & & 'aaba' \\
'ba' & & 'aba' & & 'baa' & & 'abaa' \\
'bab' & & 'abab' & & 'baaa' & \\
'bab' & & 'baab' & & 'baba' & \\
\end{align*}
\]

etc. This gives all but the first term of the Fibonacci sequence

\[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots\]
To find a formula for $F_k$, we work instead with the generating function

$$\sum_{k=0}^{\infty} F_k t^k = 1 + 2t + 3t^2 + 5t^3 + 8t^4 + 13t^5 + \cdots$$

Observe that words $w \in S_k$ correspond to paths of length $k$, starting at vertex 1 in the digraph.
Agenda

1. Motivating Problem 1 (above)

2. Counting Walks by the Transfer Matrix Method

3. Application to Problem 1

4. Counting Closed Walks

5. Counting Weighted Walks in Digraphs with Weighted Edges

6. MAPLE Worksheet for Problem 1

7. Application to Coding Theory
The Transfer Matrix Method

Let $D$ be a digraph (directed graph), possibly with loops, having vertices $1, 2, 3, \ldots, n$. Let $A = [a_{ij} : 1 \leq i, j \leq n]$ be the adjacency matrix of $D$; in other words,

$$a_{ij} = \begin{cases} 1, & \text{if } (i, j) \text{ is an edge of } D; \\ 0, & \text{otherwise}. \end{cases}$$

A walk of length $k$ in $D$ is a sequence

$$i_0 \rightarrow i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_k$$

of (not necessarily distinct) vertices such that each $i_{r-1} \rightarrow i_r$ is an edge of $D$. 
Counting Walks from $i$ to $j$

Let $w_{ij}(k)$ be the number of walks of length $k$ from vertex $i$ to vertex $j$ in $D$. Then $w_{ij}(k)$ is the $(i,j)$-entry of $A^k$. This is readily computed by reading off the coefficient of $t^k$ in the generating function $\sum_{k \geq 0} w_{ij}(k)t^k$ which in turn is the $(i,j)$-entry of

$$(I - tA)^{-1} = I + tA + t^2A^2 + t^3A^3 + \cdots.$$ 

Since the $(i,j)$-entry of $(I - tA)^{-1}$ is of the form

$$\frac{\text{poly. in } t \text{ of degree } \leq n-1}{\det(I - tA)},$$

$w_{ij}(k)$ satisfies a linear recurrence

$$w_{ij}(k + n) = \sum_{r=0}^{n-1} c_r w_{ij}(k + r) \quad \text{for all } k \geq 0$$

where $\det(I - tA) = 1 - c_{n-1}t - c_{n-2}t^2 - \cdots - c_0t^n$. The initial conditions $w_{ij}(0), w_{ij}(1), \ldots, w_{ij}(n-1)$ depend on $i$ and $j$ but the recurrence does not.
Counting All Walks

Let $w(k) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(k)$, the total number of walks of length $k$. This is the coefficient of $t^k$ in the sum of the entries of $(I - tA)^{-1}$.

In particular $w(k)$ satisfies the same recurrence as the $w_{ij}(k)$'s:

$$w(k + n) = \sum_{r=0}^{n-1} c_r w(k + r) \quad \text{for all } k \geq 0$$

but with different initial conditions.
Counting Closed Walks

Let $w_{\text{closed}}(k) = \sum_{i=1}^{n} w_{ii}(k)$, the total number of closed walks of length $k$ (i.e. starting and ending at the same vertex). This is the coefficient of $t^k$ in $\text{trace}((I - tA)^{-1})$.

In particular $w_{\text{closed}}(k)$ satisfies the same linear recurrence as the $w_{ij}(k)$’s and $w(k)$, but again with different initial conditions.

Here we assumed the initial/final vertex to be distinguished, i.e. the walks $(i_0, i_1, i_2, \ldots, i_k)$ and $(i_1, i_2, \ldots, i_k, i_0)$ are counted as distinct unless all $i_0 = i_1 = \cdots = i_k$. 
Example

Let $F_k$ be the number of ‘words’ of length $k$ consisting of ‘a’ s and ‘b’ s, with no two consecutive ‘b’ s.

\[
\begin{align*}
F_0 &= 1 & F_1 &= 2 & F_2 &= 3 & F_3 &= 5 & F_4 &= 8 \\
\text{‘} & \text{‘a’} & \text{‘aa’} & \text{‘aaa’} & \text{‘aaaa’} \\
\text{‘b’} & \text{‘ab’} & \text{‘aab’} & \text{‘aaab’} & \\
\text{‘ba’} & \text{‘aba’} & \text{‘aaba’} & & \\
\text{‘baa’} & \text{‘abaa’} & & & \\
\text{‘bab’} & \text{‘abab’} & & & \\
\text{‘baba’} & & & & \\
\end{align*}
\]

etc. This gives all but the first term of the Fibonacci sequence

\[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots\]
Observe that $F_k$ is the number of paths of length $k$, starting at vertex 1 in the digraph

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(I - tA)^{-1} = \frac{1}{1-t-t^2} \begin{bmatrix} 1 & t \\ t & 1-t \end{bmatrix}$$
\[
\sum_{k \geq 0} F_k t^k = \text{sum of (1, 1)- and (1, 2)-entries of } (I - tA)^{-1} \\
= \frac{1 + t}{1 - t - t^2} \\
= \frac{1}{\sqrt{5}} \left( \frac{\alpha^2}{1 - \alpha t} - \frac{\beta^2}{1 - \beta t} \right) \\
= \frac{1}{\sqrt{5}} \sum_{k \geq 0} (\alpha^{k+2} - \beta^{k+2}) t^k 
\]

where \( \alpha = (1 + \sqrt{5})/2, \beta = (1 - \sqrt{5})/2 \). From 
\[(1 - t - t^2) \sum_{k \geq 0} F_k t^k = 1 + t \] we obtain

\[
F_k = \begin{cases} 
1, & \text{if } k = 0; \\
2, & \text{if } k = 1; \\
F_{k-1} + F_{k-2}, & \text{if } k \geq 2 
\end{cases}
\]

so by induction, \( F_k \) is the \((k + 1)\)st Fibonacci number. From the series expansion we obtain the explicit formula

\[
F_k = \frac{\alpha^{k+2} - \beta^{k+2}}{\sqrt{5}} \quad \text{for } k \geq 0.
\]
Let $L_k$ (for $k \geq 0$) be the number of ‘words’ of length $k$ consisting of ‘a’ and ‘b’ with no consecutive ‘b’’s, and which do not both start and end with ‘b’. For technical reasons we will take $L_0 = 2$.

For $k \geq 2$, we are simply counting necklaces with amber and black beads having no two consecutive black beads; however, each necklace has a distinguished starting point (a knot in its cord) and a distinguished direction (clock-wise or counter-clockwise).

<table>
<thead>
<tr>
<th>$L_1 = 1$</th>
<th>$L_2 = 3$</th>
<th>$L_3 = 4$</th>
<th>$L_4 = 7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>‘a’</td>
<td>‘aa’</td>
<td>‘aaa’</td>
<td>‘aaaa’</td>
</tr>
<tr>
<td>‘ab’</td>
<td>‘aab’</td>
<td>‘aaab’</td>
<td></td>
</tr>
<tr>
<td>‘ba’</td>
<td>‘aba’</td>
<td>‘aaba’</td>
<td></td>
</tr>
<tr>
<td></td>
<td>‘baa’</td>
<td>‘abaa’</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>‘abab’</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>‘baaa’</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>‘baba’</td>
<td></td>
</tr>
</tbody>
</table>
These are the familiar *Lucas numbers* which satisfy the same recurrence relation as the Fibonacci numbers, but a different initial condition.

Note that $L_k$ is the number of closed walks of length $k$ in our digraph.

\[ \sum_{k \geq 0} L_k t^k = \text{trace}((I - tA)^{-1}) \]

\[ = \frac{2 - t}{1 - t - t^2} \]

\[ = \frac{1}{1 - \alpha t} + \frac{1}{1 - \beta t} \]

\[ = \sum_{k \geq 0} (\alpha^k + \beta^k) t^k \]

From $(1 - t - t^2) \sum_{k \geq 0} L_k t^k = 2 - t$ we obtain

\[ L_k = \begin{cases} 
2, & \text{if } k = 0; \\
1, & \text{if } k = 1; \\
L_{k-1} + L_{k-2}, & \text{if } k \geq 2
\end{cases} \]
From the series expansion we obtain the explicit formula

\[ L_k = \alpha^k + \beta^k \quad \text{for } k \geq 0. \]
Counting Walks with Weighted Edges

As before, $D$ is a digraph (directed graph), possibly with loops, having vertices $1, 2, 3, \ldots, n$. Assign a weight to each edge:

\[
\begin{array}{c}
\text{i} \\
\xrightarrow{a_{ij}} \text{j}
\end{array}
\]

(Non-edges have weight zero.) Define the weight of a walk

\[
\begin{array}{cccccc}
i_0 & \xrightarrow{a_{i_0i_1}} & i_1 & \xrightarrow{a_{i_1i_2}} & i_2 & \cdots & \xrightarrow{a_{i_{k-1}i_k}} & i_k
\end{array}
\]

of length $k$ to be the product

\[
a_{i_0i_1}a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{k-1}i_k}.
\]

Let $A = [a_{ij} : 1 \leq i, j \leq n]$. 
Then
\[ w_{ij}(k) := \begin{cases} \text{The sum of all weights of walks in } D \text{ of length } k \\ \text{from vertex } i \text{ to vertex } j \end{cases} \]

\[ = (i, j)\text{-entry of } A^k \]

has generating function \( \sum_{k \geq 0} w_{ij}(k)t^k \) equal to the \((i, j)\)-entry of
\[ (I - tA)^{-1} = I + tA + t^2A^2 + t^3A^3 + \cdots \]
as before.

**Example**

We have determined the number \( F_k \) of words of length \( k \) consisting of ‘a’s and ‘b’s, with no two consecutive ‘b’s. How many such words contain \( r \) ‘a’s and (therefore) \( k-r \) ‘b’s?
\[ A = \begin{bmatrix} a & b \\ a & 0 \end{bmatrix} \]

\[(I - tA)^{-1} = \frac{1}{1 - at - abt^2} \begin{bmatrix} 1 & bt \\ at & 1 - at \end{bmatrix} \]

The sum of the \((1,1)\)- and \((1,2)\)-entries is

\[
\frac{1 + bt}{1 - at - abt^2} = 1 + (a + b)t + (a^2 + 2ab)t^2 + (a^3 + 3a^2b + ab^2)t^3 + (a^4 + 4a^3b + 3a^2b^2)t^4 + \cdots
\]

Thus, for example, among the \(F_4=8\) words of length 4,

- 1 has 4 ‘a’s and 0 ‘b’s;
- 4 have 3 ‘a’s and 1 ‘b’;
- 3 have 2 ‘a’s and 2 ‘b’s.
Codes from Finite Geometry

Consider the *projective plane of order 2*:

The *binary code* of this geometry is the subspace \( \mathcal{C} \leq F^7 \) (where \( F = \{0, 1\} \mod 2 \)) spanned by the lines:

\[
\mathcal{C} = \{0000000, 1111111, 1101000, 0010111, 0110100, 1001011, 0011010, 1100101, 0001101, 1110010, 1000110, 0111001, 0100011, 1011100, 1010001, 0101110\}
\]

\(|\mathcal{C}| = 2^4; \quad \dim \mathcal{C} = 4\)
The code above is the 1-error correcting binary Hamming code of length 7.

The projective plane is constructed from $F^3$ by taking as points and lines the 1- and 2-dimensional subspaces of $F^3$.
Let $F$ be the field of order $p^e$, $p$ prime. Projective $n$-space over $F$ has as its points, lines, etc. the subspaces of $F^{n+1}$ of dimension 1, 2, etc.

**Problem:** Compute the dimension of the code $C = C_{n,p,e,k}$ spanned by the subspaces of codimension $k$.

**Solution by Hamada’s Formula** (the following theorem) is usually computationally infeasible.
Solution by the Transfer Matrix Method

**Theorem** (Bardoe and Sin, 1999)
Define $M(t) = (1 + t + t^2 + \cdots + t^{p-1})^{n+1}$.

Let $D = D_{n,p,e,k}$ be the digraph with vertices $1, 2, \ldots, k$, and the edge from vertex $i$ to vertex $j$ has weight equal to the coefficient of $t^{pj-i}$ in $M(t)$. Then

$$\dim C_{n,p,e,k} = 1 + \left( \text{sum of weights of closed walks of length } e \text{ in } D \right)$$

$$= 1 + \left( \text{coeff. of } t^e \text{ in } \text{tr}[(I - tA)^{-1}] \right)$$

where $A$ is the $k \times k$ matrix whose $(i, j)$-entry is the weight of edge $(i, j)$ (defined above).
**Example: Projective Plane of Order 2**

$\mathcal{C} =$ binary code spanned by the seven lines (subspaces of codimension $k = 1$)

$$M(t) = (1 + t)^3 = 1 + 3t + 3t^2 + t^3$$

$A = [3]$ (coefficient of $t^1$ in $M(t)$)

$$(I - tA)^{-1} = \left[ \frac{1}{1 - 3t} \right]$$

$$tr[(I - tA)^{-1}] = \frac{1}{1 - 3t} = 1 + 3t + 9t^2 + 27t^3 + \cdots$$

$$\dim \mathcal{C} = 1 + 3 = 4$$