Instructions:
- Show all your work, using the space provided on the exam. Use correct mathematical notation.
- Clearly mark your solution by circling it.
- A basic scientific calculator is allowed. No graphing calculator is allowed, and cell phones may not be used during the exam.
- Present your Photo I.D. when turning in your exam.
- The exam has 8 pages. Please check to see that your copy has all the pages.

1. (6 points) Let \( f(x) = \frac{3}{x} \). Calculate the left and right Riemann Sum, with \( n = 4 \) on \([1, 5]\).

\[
R_4 = \Delta x \cdot f(1) + \Delta x \cdot f(3) + \Delta x \cdot f(4) + \Delta x \cdot f(5) \\
= 1 \cdot \frac{3}{2} + 1 \cdot \frac{3}{3} + 1 \cdot \frac{3}{4} + 1 \cdot \frac{3}{5} = 3.85.
\]

\[
L_4 = \Delta x \cdot f(1) + \Delta x \cdot f(2) + \Delta x \cdot f(3) + \Delta x \cdot f(4) \\
= 1 \cdot \frac{3}{4} + 1 \cdot \frac{3}{5} + 1 \cdot \frac{3}{6} + 1 \cdot \frac{3}{4} = 6.25
\]
2. (16 points) Let \( g(x) = 3x^4 - 2x^3 + 1 \).

(a) Find the critical points, if any of \( g(x) \).

\[
g'(x) = 12x^3 - 6x^2 = 6x^2(2x-1).
\]

Setting \( g'(x) = 0 \) and solving for \( x \), we have the critical points

\[ x = 0, \quad x = \frac{1}{2}. \]

(b) Identify all intervals in which \( g(x) \) is decreasing.

Use the critical points in part (a) to divide the \( x \)-axis into intervals:

\[
-\infty < x < 0 \quad 0 < x < \frac{1}{2} \quad \frac{1}{2} < x < \infty
\]

\( (-\infty, 0) : g'(x) < 0 \)
\( (0, \frac{1}{2}) : g'(x) < 0 \)
\( (\frac{1}{2}, \infty) : g'(x) > 0 \)

So, \( g(x) \) is decreasing in \( (-\infty, 0) \) and \( (0, \frac{1}{2}) \).

(c) Identify all intervals in which \( g(x) \) is concave up.

\[
g''(x) = \left[ 12x^3 - 6x^2 \right]' = 36x^2 - 12x = 12x(3x-1).
\]

Setting \( g''(x) = 0 \) and solving for \( x \), we have \( x = 0, \quad x = \frac{1}{3} \)

\[
-\infty < x < 0 \quad 0 < x < \frac{1}{3} \quad \frac{1}{3} < x < \infty
\]

\( (-\infty, 0) : g''(x) < 0 \)
\( (0, \frac{1}{3}) : g''(x) > 0 \)
\( (\frac{1}{3}, \infty) : g''(x) < 0 \)

So, \( g(x) \) is concave up on \( (-\infty, 0) \cup (\frac{1}{3}, \infty) \).

(d) Fill in the blanks with the correct values.

(i) How many local extrema are there?

1

(ii) The local minimum(s) are at \( x = \frac{1}{2} \)

(iii) The absolute maximum value \[ \text{None} \]
3. (12 points) Sketch a continuous function $f(x)$ defined on $(-\infty, \infty)$ with the given properties

- $f'(x) > 0$ on $(-\infty, -1)$
- $f'(x) < 0$ on $(2, \infty)$
- $f'(-1) = 0$
- $f'(2)$ doesn’t exist.
- $f''(x) > 0$ on $(-\infty, -3)$
- $f''(-3) = f''(0) = 0$
- $f''(x) > 0$ on $(4, \infty)$
4. (12 points) A wire of length 100 inches needs to be cut so that one piece can be bent into a square and the other piece can be bent into a circle.

(a) If we used the entire wire to make a square, would that have a larger area than if we used the entire wire to make a circle? Explain.

\[
\text{Area of the square is } \left(\frac{100}{4}\right)^2 = 625 \text{ (in}^2) \text{.}
\]

The circumference of the circle is 100. So, \(2\pi r = 100\) and \(r = \frac{50}{\pi}\). Area of the circle is \(\pi \left(\frac{50}{\pi}\right)^2 = 795.77 \text{ (in}^2)\).

(b) Let \(x\) be the amount of the wire used to construct the square. Construct a function \(A(x)\) in terms of \(x\) that represents the combined area of both the square and the circle.

\[
\text{Circumference of the circle is } 100-x. \text{ So, } 2\pi r = 100-x. \text{ and hence } r = \frac{100-x}{2\pi}. \text{ Therefore,}
\]

\[
A(x) = A_{\text{square}} + A_{\text{circle}} = \left(\frac{x}{4}\right)^2 + \pi \left(\frac{100-x}{2\pi}\right)^2 = \frac{x^2}{16} + \frac{(100-x)^2}{4\pi}.
\]

(c) What is the domain of \(x\) for this problem (the interval of interest)?

\[
\text{Domain of } x \text{ is } [0, 100].
\]

(d) What is \(x\) when the combined area of the square and the circle is minimized?

Use the closed interval method to answer this question.

\[
\text{Step } #1. \text{ Find critical points in } (0, 100).
\]

\[
A'(x) = \frac{x}{8} + \frac{(100-x)(-1)}{2\pi}
\]

Set \(A'(x) = 0\). Then critical point is

\[
x = \frac{400}{4+\pi} \approx 56.01.
\]

\[
\text{Step } #2. \text{ Evaluate function values at critical point } x \text{ and at the two end points.}
\]

\[
A(56.01) = 350.06, \quad A(0) = 795.77, \quad A(100) = 625.
\]

\[
\text{Step } #3. \text{ When } x = 56.01 \text{ (in), } A \text{ is minimized.}
\]
5. (8 points) (a) Write the equation of a line that represents a linear approximation of \( h(x) = \sqrt{x} \) at \( x = 16 \).

The tangent line at \( x = 16 \) to the graph of \( y = \sqrt{x} \) is:

\[
y - 4 = \frac{1}{8} (x-16) \implies y = 4 + \frac{1}{8} (x-16).
\]

(b) Use your work above to approximate \( \sqrt{19} \). Either express your answer as a fraction or round your answer to three decimal places.

When \( x \approx 16 \), we have:

\[
\sqrt{x} \approx 4 + \frac{1}{8} (x-16).
\]

So, \( \sqrt{19} \approx 4 + \frac{1}{8} (19-16) = \frac{35}{8} \).

6. (10 points) Determine whether the Mean Value Theorem applies to the given functions on the given intervals \([a, b]\). If it applies, find a \( c \) guaranteed to exist by the Mean Value Theorem. If it does not apply, explain in words why it does not apply.

(a) \( f(x) = \frac{1}{x+1} \) on \([-2, 2]\)

\( f(x) \) is not continuous at \( x = -1 \). So the Mean Value Theorem cannot be applied to this function on \([-2, 2]\).

(b) \( g(x) = \sqrt{x} \) on \([0, 8]\)

\( g(x) \) is continuous on \([0, 8]\) and \( g'(x) = \frac{1}{3x^{2/3}} \) exists on the open interval \((0, 8)\). So the Mean Value Theorem can be applied. To find a \( c \) guaranteed by the theorem, solve

\[
\frac{1}{3x^{2/3}} = \frac{\sqrt{8} - \sqrt{0}}{8 - 0} \implies x = \frac{8}{3\sqrt{3}}.
\]
Find a function \( f \) that satisfies the following conditions

7. (6 points) \( f'(x) = \sec^2 x + \sin x; \ f(0) = 2 \)

\[
\begin{align*}
\int f(x) = \int (\sec^2 x + \sin x) \, dx &= \tan x - \cos x + C. \\
\end{align*}
\]

\[
\begin{align*}
f(0) = 2 &\implies \tan 0 - \cos 0 + C = 2 \implies C = 3. \\
\therefore \ f(x) &= \tan x - \cos x + 3. \\
\end{align*}
\]

8. (6 points) \( f''(x) = 6x; \ f'(0) = 1; \ f(0) = 1 \)

\[
\begin{align*}
f''(x) &= \int 6x \, dx = 3x^2 + C. \\
\implies f'(0) = 1 &\implies 3(0)^2 + C = 1 \implies C = 1. \\
\therefore \ f'(x) &= 3x^2 + 1. \\
\end{align*}
\]

\[
\begin{align*}
f(x) &= \int (3x^2 + 1) \, dx = x^3 + x + D. \\
\implies f(0) = 1 &\implies 0^3 + 0 + D = 1 \implies D = 1. \\
\therefore \ f(x) &= x^3 + x + 1. \\
\end{align*}
\]

For problems 9 to 11, use analytical methods to evaluate the following limits. Numerical or graphical methods will receive little credit.

9. (6 points) \( \lim_{x \to 0} \frac{e^{3x} - 1}{\sin(4x)} \) \( \text{indeterminate of type } \frac{0}{0} \).

\[
\begin{align*}
\lim_{x \to 0} \frac{e^{3x} - 1}{\sin(4x)} &= \lim_{x \to 0} \frac{\left(e^{3x} - 1\right)'}{\left(\sin(4x)\right)'} \\
&= \lim_{x \to 0} \frac{3e^{3x}}{4\cos(4x)} \\
&= \frac{3e^{3\cdot0}}{4\cos(4\cdot0)} = \frac{3}{4}.
\end{align*}
\]
10. (6 points) \( \lim_{x \to \infty} \frac{e^x - x - 1}{5x^2} \) indeterminate of type \( \frac{\infty}{\infty} \).

\[
\lim_{x \to \infty} \frac{e^x - x - 1}{5x^2} = \lim_{x \to \infty} \frac{e^x - x - 1}{5x^2} = \lim_{x \to \infty} \frac{e^x - 1}{10x}.
\]

\[
= \lim_{x \to \infty} \frac{e^x - 1}{10x} = \frac{\infty}{10} = \infty.
\]

11. (8 points) \( \lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x \) indeterminate limit of type \( 1^\infty \).

\[
\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x = \lim_{x \to \infty} e^{\ln \left(1 + \frac{2}{x}\right)^x} = \lim_{x \to \infty} e^{\ln (1 + \frac{2}{x})}.
\]

Compute \( \lim_{x \to \infty} x \ln (1 + \frac{2}{x}) = \lim_{x \to \infty} \frac{\ln (1 + \frac{2}{x})}{\frac{1}{x}} \)

\[
= \lim_{t \to 0} \frac{\ln (1 + 2t)}{t} = \lim_{t \to 0} \frac{\ln (1 + 2t)}{2t} = \frac{2}{2(1 + 2t)} = 2.
\]

\[
\therefore \lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x = e^{\lim_{x \to \infty} x \ln (1 + \frac{2}{x})} = e^2.
\]
12. (12 points) Using only the fact that \( \int_{-3}^{1} f(x)dx = -2, \int_{-3}^{1} g(x)dx = 3, \) and \\
\( \int_{1}^{4} f(x)dx = -4, \) evaluate the following integrals.

(a) \( \int_{-3}^{1} (2f(x) + 3g(x))dx \)

\[
= 2 \int_{-3}^{1} f(x)dx + 3 \int_{-3}^{1} g(x)dx \\
= 2(-2) + 3(3) = 5.
\]

(b) \( \int_{-3}^{4} \frac{1}{2} f(x)dx \)

\[
= \frac{1}{2} \int_{-3}^{4} f(x)dx = \frac{1}{2} \left[ \int_{-3}^{1} f(x)dx + \int_{1}^{4} f(x)dx \right] \\
= \frac{1}{2} \left[ -2 - 4 \right] = -3.
\]

(c) \( \int_{1}^{4} f(x)dx \)

\[
= -\int_{1}^{4} f(x)dx = -(4) = 4.
\]

For Instructor Use Only:

<table>
<thead>
<tr>
<th>Question</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Points</td>
<td>6</td>
<td>16</td>
<td>12</td>
<td>12</td>
<td>8</td>
<td>10</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>8</td>
<td>12</td>
<td>100</td>
</tr>
<tr>
<td>Score</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>