Course Note 3: Mixed, Continuous and Correlated strategies

Example 5 The Policing Game
Suppose that the police can either monitor or not monitor the citizens in a neighborhood. The police prefers not to monitor because it costs them $5. However, if the citizens know that they are not being monitored they will commit crime without getting caught and earn $10. If they commit crime and get caught, they get -5 and the police gets 8 minus their cost of 5, so the end up with 8-5=3. If the citizens do not do crime they get zero.

<table>
<thead>
<tr>
<th></th>
<th>Citizens</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Police</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Monitor</td>
<td>3,-5</td>
<td>-5,0</td>
</tr>
<tr>
<td>Do Not Monitor</td>
<td>0,10</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Equilibrium
Moving around in the normal form shows that there is no equilibrium where each of the players uses a specific strategy. If the police chooses Monitor, the citizens will choose No Crime, but then the police prefers Do Not Monitor. Then the citizens prefer Crime and then the police prefers Monitor, etc. This game therefore lacks an equilibrium in pure strategies – where “pure” means that each player chooses a specific strategy. However, there is a Nash Equilibrium in mixed strategies. A mixed strategy means that a player randomizes over at least two of her pure strategies and puts a non-zero probability on each of these strategies.

When you look for a mixed strategy, the key requirement is that the player is only willing to mix if she is indifferent between the pure strategies she mixes over. If the payoff to one of the pure strategies was less than the payoff to one of the other strategies, she would put probability zero on the former strategy instead of mixing. Thus, her statistically expected payoffs must be the same for all the strategies she mixes over.

Consider first the police. If the citizens choose No Crime, the police prefers Do Not Monitor. If the citizens choose Crime, the police prefers Monitor. So if the citizens play a pure strategy, the police will not want to mix. Assume instead that the citizens choose Crime with probability $p$ and No Crime with probability $1-p$. The police is now willing to mix – indifferent between their pure strategies- if
Expected police payoff to Monitor = Expected police payoff to Do Not Monitor
\[ p(3)+(1-p)(-5) = p(0)+(1-p)(0), \]
\[ p = \frac{5}{8} \]

By similar reasoning, the citizens are unwilling to mix if the police plays a pure strategy. However, if the police monitors with probability \( q \), the citizens are willing to mix if
\[ q(-5)+(1-q)(10) = q(0)+(1-q)(0), \]
\[ q = \frac{2}{3} \]

The only Nash Equilibrium is therefore the following mixed strategy Nash equilibrium:

**E:** \{\text{(Monitor with probability 2/3, Do not Monitor with probability 1/3), (Crime with Probability 5/8, No Crime with probability 3/8)}\}

or simply

\{\text{Monitor with probability 2/3, Crime with Probability 5/8}\}

**Interpretation of mixing**
There are several ways to interpret mixing. First, the players may genuinely choose a random strategy. For example, airport security checks and tax audits are random. Second, the players may be randomly selected from a large group of players of different ‘types’. For example, a policeman in the game above may be chosen from a large group of police academy students and among these students precisely 2/3 believe that monitoring is the best way to reduce crime. If the citizens do not observe the policeman’s type, they cannot predict his behavior. Alternatively, there may be only one policeman, but with probability 1/3 he will become a “busy policeman” preoccupied with other assignments, and have a high opportunity cost of monitoring, while with probability 2/3 he will have plenty of time and always monitor.

**Fragility of mixing**
In a mixed Nash equilibrium like the one in the policing game, the players are willing to mix because they are indifferent between their pure strategies (or at least the ones they mix over). However, if you are indifferent between your strategies, you suffer no payoff loss from changing the probability you put on each. For example, in order to make the citizens willing to mix, the policeman must monitor with precisely probability 2/3. But the policeman is indifferent to making shifting the Monitor probability to, for example, 0.5. There is not a perfect answer to how the mixed equilibrium is sustained, but if we use the ‘unknown type’ interpretation of mixed strategy equilibrium, there is no random behavior on the individual level. That way we put the randomness on nature rather than
the players. Another, evolutionary argument is that the players may come in “generations” with a constant propensity to choose Monitor or Do Not Monitor. Then a process similar to natural selection may lead to a stable population mix that behaves like a mixed strategy equilibrium. For example, evolutionary dynamics could lead to a population mix where 2/3 of the police force are type “Monitor” and 1/3 are type “Do Not Monitor” and likewise 5/8 of citizens are type “Criminal” and 3/8 type “Honest”. We will cover this type of evolutionary game theory later on (or see Rasmusen Ch. 5.6). A third explanation may use some repeated game logic where, if a player deviates from the mixed strategy in the current period, the other player will deviate in future periods (it is complicated, though, because in the policing game at least, the future periods still only have the same mixed Nash equilibrium as the current period). We will also look at repeated games later.

Is the policing game realistic? A variant

Before we assumed that the citizens and the police moved simultaneously. Then the only equilibrium was in mixed strategies. If we compute the equilibrium expected payoffs by substituting the \((p,q) = (5/8,2/3)\) values into the payoff functions, we find that the police gets \(q[p(3)+(1-p)(-5)] + (1-q)[p(0)+(1-p)(0)] = 0\). We can similarly calculate that the citizens get \(p[q(-5)+(1-q)(10)] + (1-p)[q(0)+(1-q)(0)] = 0\). There is also an easier way to calculate this, which is to first observe that each player gets the same payoff from all the strategies she mixes over. The total expected payoff therefore equals the payoff to any particular strategy. For example, we could simple compute the policeman’s expected payoff as the expected payoff to monitoring with probability one, \(p(3)+(1-p)(-5) = 0\) when \(p=5/8\). The citizens’ total expected is similarly just their expected payoff to, say, No Crime, which is \(q(0)+(1-q)(0) = 0\).

The citizens can actually do better if they are able to commit to “Crime with probability \(5/8 - \varepsilon\)” for \(\varepsilon\) small but positive. For example, the citizens could organize a community group where members monitor each other to keep the Crime probability just below 5/8. Since the probability of Crime is less than 5/8, the police will prefer not to monitor. Therefore, the citizens will get payoff \((5/8 - \varepsilon)(10) + (3/8 + \varepsilon)(0) \approx 6.25 > 0\). (Rasmusen chapter 3 has a similar example, where the tax authority commits to a tax audit probability).

Notice, however, that we have now changed the structure of the game: it is now a sequential move game, where in the first stage of the game the citizens choose a Crime probability and in the second stage the police choose whether to Monitor. The game tree for this new game is shown below. The arch for citizens’ choice is used to show that the Crime probability is a continuous choice variable – otherwise we would need infinitely many branches, one for each possible probability. You can think of the left and right branches at first in the game as representing Crime probability 0 and 1, respectively. Which description – the simultaneous move or the sequential move game - is most realistic depends on the situation, but the example shows the timing assumption (simultaneous versus sequential moves) is important and that, with respect to terminology, choosing a random strategy can be different from choosing a mixed strategy.
Example 6 Cournot competition and continuous strategies

In a Cournot game between firms selling related (homogenous or heterogeneous) goods, the firms simultaneously choose the quantities they want to produce. When a firm produces more, the price falls, typically reducing the incentive for the other firm to produce more. This is a simultaneous-move game with continuous strategies (quantities produced). Rasmusen Section 3.5. and most microeconomic theory or industrial organization textbooks have examples. Games with Bertrand competition – where firms compete on price instead of quantity – are also simultaneous-move, continuous strategy games. Another example is a Stackelberg game where first one firm chooses quantity or price, then the other observes the move and chooses quantity or price also.

Example 7 Chicken, mixed and correlated strategies

The classic game of chicken has two drivers racing against each other. A player who veers off first is a “chicken” and makes the other the winner. However, if the players collide, both suffer a big loss.

Normal form

<table>
<thead>
<tr>
<th>Player 2</th>
<th>C</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continue (C)</td>
<td>-10,-10</td>
<td>10,-5</td>
</tr>
<tr>
<td>Veer off (V)</td>
<td>-5,10</td>
<td>0,0</td>
</tr>
</tbody>
</table>
Nash Equilibriums:
You can check the game has two pure strategy equilibriums:

E1: {C,V}

E2: {V,C}

Are there also mixed strategy equilibriums? Suppose that player 2 plays Continue with probability $p$ and therefore Veer off with probability $(1-p)$. Player 1 is willing to mix between his strategies if and only if

Expected player 1 payoff to Continue = Expected player 1 payoff to Veer off

$p(-10) + (1-p)(10) = p(-5) + (1-p)(0),$

$p = \frac{2}{3}$

and by symmetry, we can see that player 1’s probability $q$ of playing Continue, in order for player 2 to be willing to mix, must be $q = \frac{2}{3}$.

So we also have

E3: {Continue with probability 2/3, Continue with probability 2/3}

Correlated Equilibrium
A correlated equilibrium is a mixed strategy equilibrium\(^1\) where (i) each player’s actual strategy choice depends on a signal she receives from a “public randomization device” prior to the game; (ii) players’ signals are correlated; (iii) once the signals are received, players pick optimal strategies given their information, including their rational (Bayesian) beliefs about the signals received by other players.

In Chicken, such a public correlation device can be useful because it avoids the occasional collision. In particular, notice that under the mixed strategies, each player’s expected payoff is

$$\frac{2}{3}[(\frac{2}{3})(-10)+(\frac{1}{3})(10)] + \frac{1}{3}[(\frac{2}{3})(-5)+(\frac{1}{3})(0)] = -\frac{30}{9} = -\frac{10}{3}$$

On the other hand, suppose that the players use the outside temperature as a public randomization device. With temperature $X$ or higher, player 1 plays Continue and player 2 Veers off and opposite for temperature below $X$. Notice that these strategies are optimal for both players after any realization of the random variable – as they should be in

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\(^1\) A pure strategy equilibrium is technically a special case of a mixed strategy equilibrium; your mixing probabilities are 1 on some strategy and zero on other strategies. So when we say ‘mixed equilibrium’ this could be a pure strategy equilibrium; ‘strictly mixed’ denotes that all strategies of all players have positive probability of being played, ruling out pure strategies.
correlated equilibrium. If for example X is chosen to have probability 0.5 of being at or above the current temperature, both players’ expected payoff is now

\[(0.5)(10)+(0.5)(-5) = 2.5\]

and the equilibrium is

E4: \{((Continue if the temperature is at least X; otherwise, veer off), (Continue if the temperature is below X; otherwise, veer off))\}.

Sometimes players can also do without actually having the public randomization device available. For example, the strategies \{Continue, Veer off\} may be a focal point or, if the players play repeatedly, they can take turns playing Continue and Veer off.

Example 8 A game where imperfect correlation is preferable
In the correlated equilibrium in Chicken we just described, by choice of X players can choose (Continue, Veer off) anywhere from 0-100% of the time and (Veer off, Continue) the rest of the time. Either way, however, they get a joint payoff of 5. We can express this graphically by saying players can achieve any payoffs in the \textit{convex hull of the pure strategy equilibrium payoffs}: the convex hull of the pure strategy equilibrium payoffs is the smallest convex set containing all the pure strategy equilibrium payoffs. It is shown below

![Graph](image_url)

However, consider now the simultaneous move game with normal form matrix:

**Normal form**

<table>
<thead>
<tr>
<th></th>
<th>Player 2 ( (L) )</th>
<th>Player 2 ( (R) )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Player 1</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Up ( (U) )</td>
<td>5,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Down ( (D) )</td>
<td>4,4</td>
<td>1,5</td>
</tr>
</tbody>
</table>
Nash Equilibriums (so no correlation)
Without correlation this game who two pure strategy equilibriums and one mixed equilibrium:

E1: {U,L}

E2: {D,R}

E3: {{(U with probability 0.5, D with probability 0.5), (L with probability 0.5, R with probability 0.5)}

E1 gives a total expected payoff of 6 (5 for player 1, 1 for player 2), as does E2, while E3 gives expected payoff 2.5 to each player or a total of 5. Notice for later use that the highest total payoff across the players is 6 or 3 per player on average. We will now show that the average payoff can increase if we use a correlated equilibrium.

Correlated equilibriums
Suppose first that players use a public randomization device which leads their signals to be perfectly correlated. In this case they will switch between E1 and E2 and always get payoffs (1,5) or (5,1), putting the average on in the convex hull of the set of pure strategy equilibrium payoffs (see the figure above). The average payoff per player is still 6/2 = 3.

E4: {{(Play Up if temperature is at least X; otherwise play Down), (play Left if temperature is at least X, otherwise play Right)}

However, suppose now that the players pick a public randomization device for which their signals are not perfectly but only partly correlated. The device works as follows: the device has three equally likely states, A, B and C. If state A occurs, then player 1 is perfectly informed about it, but if states B or C occur, player 1 only knows that the state is in the set {B,C}. (We say player 1 observes the information partition {{A},{B,C}}, that is, she will know if the event belongs to the set {A} or the set {B,C}). So player 1 cannot distinguish states B and C. On the other hand, player 2 is perfectly informed if state C occurs, but if A or B occurs, he only knows that the state is in the set {A,B}. (So he observes the information partition {{A,B},{C}}). Now consider this proposed equilibrium:

E5: {{(U if the state is A, D if it is B-C), (L if the state is A-B, R if it is C)}

We will now prove that this is a correlated equilibrium by proving that no player has incentive to deviate to another strategy in any subgame (=in any realized state).

In state A player 1 plays U and Player 2 plays L. Player 1 knows the state is A so she knows player 2 will play L. Player 2, on the other hand, cannot tell if the state is A or B, but he believes it is A with probability (using Bayes rule).
\[ p(A \mid [A, B]) = \frac{P(A \cap [A, B])}{P([A, B])} = \frac{P(A)}{P(A) + P(B)} = \frac{1/3}{1/3 + 1/3} = 0.5 \]

Player 2 therefore perceives probability 0.5 that player 1 observed state A and will play U, and probability 0.5 that player 2 observed state B and will play D. No deviation in state A then requires

Player 1: \[ \text{payoff to U} \geq \text{payoff to D} \]

Player 2: \[ \frac{0.5(1) + 0.5(4)}{0.5(0) + 0.5(5)} \geq \frac{0.5(0) + 0.5(5)}{0.5(0) + 0.5(5)}. \]

Now suppose that the state is B. Then player 1 will play D and player 2 will play L. Player 2 cannot tell if the state is A or B but believes it is A with probability 0.5 as calculated above. As before, then, he expects player 1 to play U with probability 0.5 and D with probability 0.5. Player 1 similarly believes that the state is B with probability 0.5 and C with probability 0.5. Therefore he believes player 2 will play L with probability 0.5 and R with probability 0.5. No deviation then requires

Player 1: \[ \frac{0.5(4) + 0.5(1)}{0.5(0) + 0.5(5)} \geq \frac{0.5(5) + 0.5(0)}{0.5(5) + 0.5(0)} \]

Player 2: \[ \frac{0.5(1) + 0.5(4)}{0.5(0) + 0.5(5)} \geq \frac{0.5(0) + 0.5(5)}{0.5(0) + 0.5(5)}. \]

Finally, if the state is C player 1 will play D and player 2 will play R. Player 1 cannot tell if the state is B or C and believes it is B with probability 0.5. Therefore she again believes player 2 will play L with probability 0.5 and R with probability 0.5. Player 2 knows it is C, so he knows player 1 will play D. No deviation requires

Player 1: \[ \frac{0.5(4) + 0.5(1)}{0.5(0) + 0.5(5)} \geq \frac{0.5(5) + 0.5(0)}{0.5(5) + 0.5(0)} \]

Player 2: \[ \frac{5}{\text{payoff to R}} \geq \frac{4}{\text{payoff to L}}. \]

The expected payoffs in this equilibrium are, for players 1 and 2 both,

\[ (1/3)(5) + (1/3)(4) + (1/3)(1) = 10/3 \]

and so the payoff total for both players is 20/3 > 6 and the payoff per player is 10/3 > 3.

Now recall that the most the players could get with a pure or mixed strategy Nash equilibrium, or a correlated equilibrium with perfect correlation, was 3 per player on
average. Using imperfect correlation allows them to achieve a higher average per player than anywhere in the convex hull of the pure strategy equilibrium payoffs. The key to this result is that with imperfectly correlated signals, the players are willing to do things (certain actions) they would not do if they had perfect information. In particular, in state B, the players play the cooperative (D,L), while if player 1 was sure that player 2 had observed state B and would therefore play L, she would deviate from D to U and they could not “cooperate” on D,L. Similarly, if player 2 was sure that player 1 had observed state B and would therefore play D he would deviate from L to R. By picking the right randomization device prior to playing the game the players can commit to mutually beneficial ignorance, improving their payoffs.  

Example 9 An N player Public Goods Game

Suppose that each player wins B>1 if a public good is provided. The cost of providing the public good – for example, cleaning up a city park - is 1. Each of the N players can choose simultaneously whether to pay or not pay for the public good. If at least one player contributes, the good is provided, and if more than one player contributes, the contributions above 1 are wasted. For example, by the time a player realizes that somebody else is already cleaning up the park, it is too late to go back to her regular job.

Normal form representation
Since there are N players we would need, the N by N matrix to show the strategic form. However, we can exploit the simplicity of the game (that all players face identical choices and the only actions are “pay” and “do not pay”) to state that each player i faces a

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2 Game theory has many examples where less information is better for a player. In many cases, the reason is that not having information available commits you to not exploiting it against the other player. This way you can make other players trust you and do things they would not otherwise risk doing.
matrix (notice that this is not really the normal form of the game, because the other players do not act as a group).

<table>
<thead>
<tr>
<th>All other players</th>
<th>$M&gt;0$ others pay</th>
<th>$M=0$ others pay</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Player i</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pay</td>
<td>$B-1, (N-1)B-X$</td>
<td>$B-1, (N-1)B$</td>
</tr>
<tr>
<td><em>Do not pay</em></td>
<td>$B, (N-1)B-X$</td>
<td>$0,0$</td>
</tr>
</tbody>
</table>

where $M$ is the number of others who paid for the public good. The normal form shows that player $i$ prefers to pay if and only if no other players pay. This gives $N$ pure strategy Nash equilibriums

**Ei : {Pay if you are player $i$, Do Not Pay if you are not player $i$}, i=1…N**

Alternatively, suppose that the players play the same mixed strategies, where they pay with probability $p$. $p$ must satisfy

Expected payoff if I pay = expected payoff if I do not pay

$$(1-p)^{N-1}(B-1) + (1-(1-p)^{N-1})(B-1) = (1-p)^{N-1}(0) + (1-(1-p)^{N-1})(B),$$

$$p = 1 - \left(\frac{1}{B}\right)^{\frac{1}{N-1}}$$

which is declining in $N$ and increasing in $B$, as you might also expect based on economic intuition (why?)

The total chance of the good being provided is one minus the probability that nobody pays. The probability that nobody pays is $(1-p)^N$. From just above we know that

$$p = 1 - \left(\frac{1}{B}\right)^{\frac{1}{N-1}},$$

which means that $1-p = \left(\frac{1}{B}\right)^{\frac{1}{N-1}}$. Therefore, $(1-p)^N = \left(\frac{1}{B}\right)^{\frac{N}{N-1}}$ and the probability that somebody pays is
\[ 1 - (1 - p)^N = 1 - \left( \frac{1}{B} \right)^{N-1}. \]

This probability of public goods provision is falling in \( N \). Therefore, even though the social surplus to providing the public good, \( NB-1 \), is larger when the group size \( N \) is large, provision is less likely because of the free-riding effect (waiting for the others to provide the good). [An example applying this model may be the 1964s murder of Kitty Genovese in New York City as explained in the Dixit-Skeath textbook on the syllabus. Despite the murder being observed by a large crowd of spectators (though nobody may have seen what was going on clearly, you can try googling the victim’s name), nobody acted. The failure to act may at appear puzzling or the result of crowd mentality, but the mixed Nash Equilibrium we just described provides another explanation: it predicts precisely that a high number of spectators leads to a low chance of preventative action].

**Correlated equilibrium:** can use of a public randomization device make the equilibrium more efficient? Suppose that the group simply picks a random number between 1 and \( N \) out of a hat before playing and whoever has that number pays for the public good. Then, while in the mixed strategy equilibrium the good might not be provided or more than one dollar total may be paid, with public randomization the good is always provided and only one dollar is paid, which is the efficient solution. The expected cost per player is only \( 1/N \) and the benefit is always \( B \), so the net benefit is \( B-1/N \). If you compute the expected benefit and cost in the mixed equilibrium, you find that it is \( B-1 < B-1/N \). So the correlated equilibrium would be more efficient.

**Sequential moves** suppose that the players move in turn, starting with player 1’s payment decision, then player 2’s, and so on up to player \( N \). Now, the same Nash Equilibriums as above still exist. However, most of them are not perfect Nash Equilibriums. This is easily seen by backward induction: when the last player moves, if no other player has paid for the public good she prefers to pay since \( B > 1 \). Therefore, no previous player will pay and the unique perfect equilibrium outcome is that the last player provides the public good.

**Example 10: Rock-Paper Scissors**

**Normal form**

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>( R )</td>
<td>( P )</td>
<td>( S )</td>
<td></td>
</tr>
<tr>
<td>( R )</td>
<td>0,0</td>
<td>-1,1</td>
<td>1,-1</td>
<td></td>
</tr>
<tr>
<td>( P )</td>
<td>1,-1</td>
<td>0,0</td>
<td>-1,1</td>
<td></td>
</tr>
<tr>
<td>( S )</td>
<td>-1,1</td>
<td>1,-1</td>
<td>0,0</td>
<td></td>
</tr>
</tbody>
</table>
You can check that this game has no Nash Equilibrium in Pure Strategies. However, let’s try mixed strategies where player 1 plays Rocks and Paper (R and P) with probabilities \( p_R \) and \( p_P \) and plays Scissors (S) with probability \( 1 - p_R - p_P \). Now, for player 2 to be willing to mix the payoffs to R, P and S for player 2 must be the same:

\[
\begin{aligned}
\text{Expected payoff to Rock} & \quad p_R 0 + p_P (-1) + (1 - p_R - p_P)1 = p_R 1 + p_P 0 + (1 - p_R - p_P)(-1) \\
\text{Expected payoff to Paper} & \quad p_R 1 + p_P 0 + (1 - p_R - p_P)(-1) = p_R (-1) + p_P 1 + (1 - p_R - p_P)0
\end{aligned}
\]  

\[(1)\]

\[(2)\]

This is two equations in two unknown variables, \( p_R \) and \( p_P \). We can solve them by solving equation (1) for \( p_R \):

\[
- p_P + 1 - p_R - p_P = p_R - 1 + p_R + p_P \\
- 2p_P + 1 - p_R = 2p_R - 1 + p_P \\
- 3p_P = 3p_R - 2 \Rightarrow p_P = -p_R + 2/3
\]

Equation (2) can be written \( p_R - 1 + p_R + p_P = p_R + p_P \) or \( p_R - 1 + p_P = p_P \)

Now plugging in \( p_P = -p_R + 2/3 \) gives \( p_R - 1 + (-p_R + 2/3) - 1 + p_R + p_P \).

The first equality can be written \( 1 - p_R - 2p_P = p_P - p_R \Rightarrow p_P = 1/3 \). Likewise, the second equality gives \( 2p_R - 1 + p_P = p_P - p_R \Rightarrow p_R = 1/3 \). Then the probability of scissors is \( 1 - p_R - p_P = 1/3 \). We can repeat this procedure for player 1 by assuming that player 2 plays Rocks and Paper (R and P) with probabilities \( q_R \) and \( q_P \) and plays Scissors (S) with probability \( 1 - q_R - q_P \). We then get the mixed equilibrium:

**E: \{R,P,S with probability 1/3 each; R,P,S with probability 1/3 each\}**

**Modified Rock-Paper-Scissors**

Assume the alternative Rock-Paper-Scissors payoffs below:

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
<th>R ((p_R))</th>
<th>P ((p_P))</th>
<th>S ((p_S = 1 - p_R - p_P))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 1</td>
<td>(q_R)</td>
<td>0,0</td>
<td>-1,1</td>
<td>1,1</td>
</tr>
<tr>
<td></td>
<td>(q_P)</td>
<td>1,-1</td>
<td>0,0</td>
<td>-1,1</td>
</tr>
<tr>
<td></td>
<td>(q_S)</td>
<td>-2,1</td>
<td>0,-1</td>
<td>-1,0</td>
</tr>
</tbody>
</table>
There is no Nash Equilibrium in pure strategies. Regarding mixed equilibrium, strategy S for player 1 is weakly dominated by strategy P (since P is better than S if player 2 picks R because 1 > -2). Player 1 would therefore never mix over S unless player 2 never plays R. Formally:

\[
\frac{p_R(2) + p_p(1 - p_p - p_R)(-1)}{\text{Player 1 expected payoff to S}} < \frac{p_R(1) + p_p(1 - p_p - p_R)(-1)}{\text{Player 1 expected payoff to P}}
\]

unless \( p_R = 0 \).

Therefore, there is no mixed Nash equilibrium where each player plays each strategy with positive probability. We should instead look for an equilibrium where either

**Case 1:** \( q_S = 0, p_R > 0 \): player 1 never plays S

**Case 2:** \( q_S > 0, p_R = 0 \): player 1 is willing to play S since \( p_R = 0 \)

**Case 3:** \( q_S = p_R = 0 \): player 1 never plays S and player 2 never R.

**Case 1:** \( q_S = 0 \Rightarrow q_p = 1 - q_R \).

In this case player 2 is willing to mix over R, P and S if all give the same expected payoff. However, given player 1 does not play S, for player 2 P dominates R. Therefore player 2 will not mix over R. Formally

\[
q_R 0 + (1 - q_R)(-1) < q_R 1 + (1 - q_R)0 \quad \text{for any probability} \quad q_R \in [0,1].
\]

Thus player 2 will not randomize over R is player 1 never plays S. This eliminates Case 1.

**Case 2:** \( p_R = 0 \Rightarrow p_S = 1 - p_p \).

In this case they play the following game:

<table>
<thead>
<tr>
<th>Player 2</th>
<th>( R (q_R) )</th>
<th>( P (q_p) )</th>
<th>( S (p_s = 1 - p_p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>-1, 1</td>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
<tr>
<td>( P )</td>
<td>1, -1</td>
<td>0, 0</td>
<td>-1, 1</td>
</tr>
<tr>
<td>( S )</td>
<td>-2, 1</td>
<td>0, -1</td>
<td>-1, 0</td>
</tr>
</tbody>
</table>

Player 1 is willing to mix over R, P and S if all give same expected payoff:

\[
p_p(-1) + (1 - p_p)1 = p_p 0 + (1 - p_p)(-1) = p_p 0 + (1 - p_p)(-1)
\]
Rewriting the first equality gives $1 - 2p_p = p_p - 1 \Rightarrow p_p = 2/3$. In turn, $p_s = 1 - 2/3 = 1/3$. So player 1 is willing to mix if player 2 plays P with probability 2/3 and S with probability 1/3.

Player 2 will be willing to not play R and to mix over P and S if:

$$q_R 0 + q_p (-1) + (1 - q_R - q_p) 1 \leq q_R 1 + q_p 0 + (1 - q_R - q_p) (-1) = q_R (-1) + q_p 1 + (1 - q_R - q_p) 0$$

The equality (expected payoff from P = expected payoff from S) gives $q_R = 1/3$. The inequality (expected payoff from R ≤ expected payoff from P) implies that $q_p \geq 2/3 - q_R$. Using $q_R = 1/3$ we need $q_p \geq 1/3$. We also know that $q_s = 1 - q_R - q_p = 1 - 1/3 - q_p = 2/3 - q_p$. Altogether there is a mixed Nash equilibrium:

**Mixed NE1:** { $(q_R = 1/3, q_p \geq 1/3, q_s = 2/3 - q_p; p_p = 2/3, p_s = 1/3)$ }.

**Case 3:** $p_R = 0 \Rightarrow p_s = 1 - p_p; q_s = 0 \Rightarrow q_p = 1 - q_R$

Now they effectively play the smaller game

<table>
<thead>
<tr>
<th>Player 2</th>
<th>P (p_p)</th>
<th>S (p_s = 1 - p_p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>R (q_R)</td>
<td>-1,1</td>
<td>1, -1</td>
</tr>
<tr>
<td>P (q_p = 1 - q_R)</td>
<td>0,0</td>
<td>-1,1</td>
</tr>
</tbody>
</table>

Mixing for player 1: $p_p (-1) + (1 - p_p) (1) = p_p (0) + (1 - p_p) (-1) \Leftrightarrow p_p = 2/3$

Mixing for player 2: $q_p (1) + (1 - q_p) (0) = q_p (-1) + (1 - q_p) (1) \Leftrightarrow q_p = 1/3$

This gives another mixed Nash equilibrium:

**Mixed NE2:** { $(q_R = 2/3, q_p = 1/3, q_s = 0), (p_R = 0, p_p = 2/3, p_s = 1/3)$ }