An Example of a Hypothetical Learning Progression:
The Emergence of Informal Mathematical Induction

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Some time ago, I asked Nikki, a kindergartner, what she thought the largest number was. The girl responded, "a million." I then asked what number she thought came after a million. After a moment's thought, Nikki responded, "a million and one." I next asked what she thought came after a million and one. Again the girl thought for a moment and answered, "a million and two." I pressed Nikki further by asking what she thought came after a million and two. After a few moments of thoughtful reflection, she concluded, "there is no largest number."

Nikki appears to use various types of mathematical reasoning—including an informal version of a relatively advanced form of reasoning—to construct the concept of *indefinite number succession*—an example of the concept of *infinity*.

- *Intuitive reasoning* entails playing a hunch—using what is obvious (appearances), what feels right (an assumption), or previous experience (precedent) to draw a conclusion. For example, when I asked Nikki what number came after a million, she probably assumed from her previous experience with questioning adults that her answer (a million is the largest number) was incorrect and that there was an even larger largest number.
- *Empirical induction* entails examining examples (particulars) and discerning a commonality or pattern (discovering a generality). For example, from the succession of questions (examples), Nikki recognized a pattern, namely that even the "largest" numbers have a number after them (induced the generality that every number has a successor).
- *Deductive reasoning* involves reasoning from a premise or premises assumed to be true (a generality or generalities) to logically arrive at a conclusion about a particular case. Asked what comes after a million (or million one), Nikki probably used her knowledge of the recursive nature of the counting sequence to respond. Specifically, she may have reasoned that, after a term starting a new series (e.g., a decade term such as *twenty* or hundred term such as *three hundred*), the rest of the series is generated by combining this first term with the single-digit sequence *one* to *nine* (e.g., after *twenty* comes *twenty-one, twenty two*…; after *three hundred* comes *three hundred one, three hundred two*; premise 1) and, if a *million* introduces a new series (premise 2), then the next terms must be a *million one*, a *million two*, and so forth. Unlike the conclusions drawn from intuitive or inductive reasoning, note that Nikki's "deduction" *necessarily* follows from what is given.
- *Mathematical induction* (reasoning by recurrence) combines both empirical induction and deductive reasoning to form a unique type of reasoning[1] (Poincaré, 1905, and Piaget, 1942, cited in Smith, 2008; Smith, 2002). Similar to empirical induction, it entails making a

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[1] In contrast, Rips and colleagues (Rips, Asmuth, & Bloomfield, 2006, 2008; Rips, Bloomfield, & Asmuth, 2008) argue from the perspective of logicism that mathematical induction is a form of deductive reasoning (necessitating), which is distinct from empirical induction (universalizing). See Smith (2002) for a critique of logicism.

generalization about a reoccurring pattern (recursive property) for an integer n and its successor ($n + 1$), but unlike empirical induction and like deductive reasoning, the conclusion about n and $n + 1$ (induced property) is necessarily true of all integers. For example, Nikki initially appeared to conclude that a million (n) was the number after the next-largest number, and if a million is the largest number, then there is no number after it. However, when she allowed that there is a number after a million, she concluded that it is not the largest number but that a million one ($n + 1$) is the largest number. But there is also a number after a million and one, namely a million two, and so $n + 1$ is not the largest number. Nikki concluded (deduced) that this process could, in principle, go on forever (i.e., for any number named, there is a successor and thus the counting sequence is infinite).

Like deductive reasoning, then, mathematical induction is creative in that it allows one to use observations and existing knowledge to create new knowledge—extend understanding beyond the limits of extant knowledge. As the case of Nikki illustrates, a primitive or informal version of mathematical induction can be an important process in young children's meaningful mathematical learning.[2] Indeed, it even enabled her to comprehend what she could not experience directly (infinity).

The aim of this article is to raise questions about the development of informal mathematical induction. Discussed, in turn, are (a) previous research on this type of mathematical reasoning; (b) a hypothetical learning progression of how the informal form of this reasoning might develop; (c) educational implications of the learning progression, and (d) issues that need to be addressed to better understand this development, construct an empirically based learning progression, and devise effective instruction for promoting this development.

Existing Research on Informal Mathematical Induction

Despite the potential importance of informal mathematical induction, little research has examined the development of this type of reasoning in early childhood. Although empirical induction and deductive reasoning have long been a concern to developmental psychologists and mathematics educators (see, e.g., Ennis, 1969; Evans, 1982; Morris, 2000; 2002), mathematical induction has largely been ignored, perhaps partly because of the assumption that children were not capable of such advanced reasoning (see, e.g., Rips, Bloomfield, & Asmuth, 2008).

Inhelder and Piaget (1963; cited by Smith, 2002) concluded from their study of (informal) mathematical induction that children aged 5 to 7 years old could make such inferences on the basis of iterative actions and that their reasoning was modal (i.e., understood their conclusions were necessarily true). In his replication and adaptation of this research, Smith (2002) came to a similar conclusion. Rips, Bloomfield, and Asmuth (2008) argued that Smith's (2002) results actually bear on universal generalization (empirical induction), not mathematical induction. Smith (2008) countered that the three questions used in his 2002 study (regarding the base

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[2] Obvious differences between informal and formal mathematical induction are that the latter requires (a) formal and advanced mathematical training and (b) the formulation of a formal or logical proof. Another possible difference is that, like Nikki, young children may need to consider more than a single case (an integer and its successor) to detect a pattern. In Nikki's case, she considered a million and a million one (first case), then a million one and a million two (second case), and was in the process of considering a third case, a million two and a million three when she had her insight regarding indefinite succession. Consistent with Poincaré's (1905) and Rips, Bloomfield, and Asmuth's (2008) criteria for mathematical induction, Smith (2002, 2003, 2008) suggested that an operational definition of informal mathematical induction should entail (a) establishing a base equality or inequality; (b) assessing universality about number; and (c) gauging the necessity about number. It seems reasonable to assume that Nikki understood that the "largest number" was larger than its predecessor, that any number would have a successor, and that this would be necessarily true of any number.

equality/inequality through serial additions, universality about number, and necessity about number) were operational matches to Poincaré and Rips et al.'s criteria for mathematical induction and that his evidence on the first two questions was significant and that on the third (necessity) was promising. In brief, Smith (2008) concluded that his evidence indicated that 5- to 7-year olds were capable of a primitive (if fallible) version of mathematical induction.

Although they did not examine informal mathematical induction directly, a few researchers (Evans, 1983; Evans & Gelman, 1982; Harnett & Gelman, 1998) found evidence that primary-level pupils comprehend the idea of indefinite succession. Specifically, Hartnett and Gelman (1998) found that the majority of their grade 2 and a quarter of their grade K participants understand that every natural number has a successor and half of the grade K pupils were classified as "waverers." In brief, the existing—albeit sparse—research indicates that children as young as 5 years of age are capable of informal mathematical induction.

Hypothetical Learning Progression

Unfortunately, to date no clear evidence exists about the development of informal mathematical induction—including its developmental precursors (Baroody, 2005). In this section, we first describe a basic assumption of a hypothetical learning progression for informal mathematical induction and then use existing developmental evidence and theory to describe a plausible progression.

Assumption. Our proposed hypothetical learning progression is based on the assumption that children typically construct knowledge in a bottom-up, not a top-down, manner. Although probably an exaggeration, Colburn (1828) observed that "it is not too bold an assertion to say, that no man ever actually learned mathematics in any other method than by analytic [empirical] induction; that is, by learning the principles by the examples he performs; and not by learning principles first, and then discovering by them how the examples are to be performed." In contrast Rips and colleagues (Rips, Asmuth, & Bloomfield, 2006, 2008; Rips, Bloomfield, & Asmuth, 2008) offered a critique of this "bootstrap" developmental view (e.g., Carey, 2004; Piantadosi, Tenenbaum, & Goodman, 2012) and suggested that number and arithmetic are built in a more top-down fashion by constructing mathematical schemas that permit mathematical inferences. One particularly relevant argument they made for rejecting experience with physical quantities as the basis for a concept of the natural numbers is that such experiences are always finite and natural numbers are infinite. However, Smith (2002) concluded that, for Piaget, mathematical induction entailed both intuition (empirical induction) and logic (deductive reasoning) and the source of the former, as the case of Nikki illustrates, is reflection on repeated mental actions (see also Sophian, 2008). For other reasons to question the top-down view and consider the bottom-up alternative, see, for example, Andres, Di Luca, and Presenti (2008); Barner (2008); Carey (2008); Cowan (2008); Smith (2008); and Sophian (2008).

Steps in the progression. The steps in the proposed hypothetical learning progression date to children's first number words.

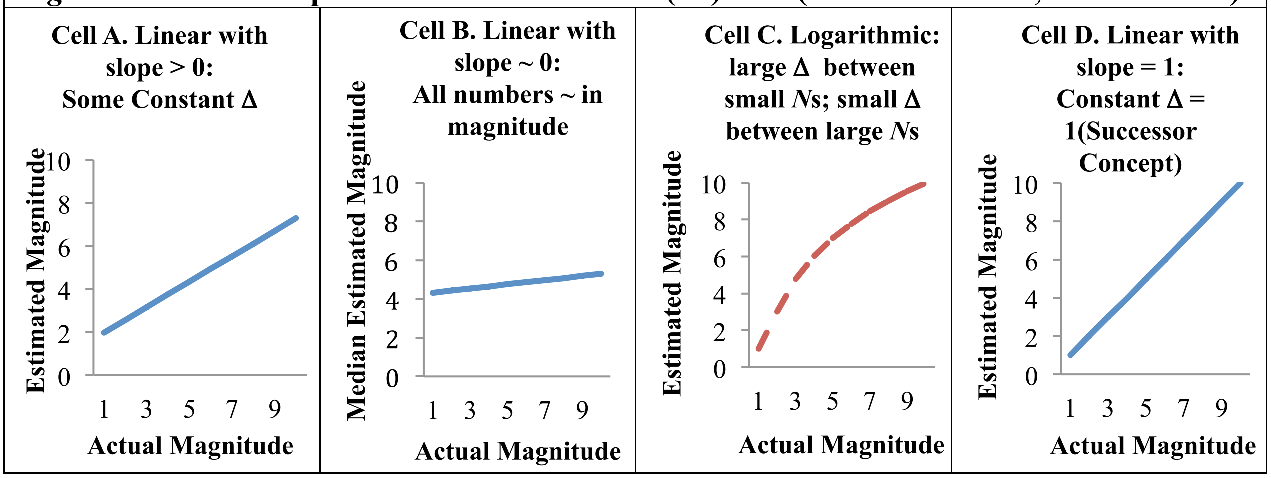
- *Verbal subitizing*. Verbal subitizing entails immediately and reliably recognizing the total number of items in small collections and labeling it with an appropriate number word (Lord, Reese, & Volkmann, 1949). Although preverbal capacities (e.g., a basic understanding of singular and plural instances) provide a basis for constructing a number concept, symbolic tools such as language permit children to construct an exact concept of cardinal numbers (Mix, Sandhofer, & Baroody, 2005; Palmer & Baroody, 2011; Spelke, 2003). As with other words, children seem to first use "two" and "three" with little specific meaning or to indicate "many" instead of a specific quantity (Mix, Huttenlocher, & Levine, 2002; Sarnecka, Kamenskaya, Yamana, Ogura, & Yudovina, 2007; von Glasersfeld, 1982; Wagner & Walters, 1982; Wynn, 1992). As with many other concepts, children may construct number concepts via an inductive process involving visual examples and non-examples. As children see a number word associated with various visual examples (e.g., "*two* eyes," "*two* hands," "*two* socks," "*two* shoes," "*two* cars") but not non-examples (e.g., "take one cookie, not two," "that's five fingers, not two"), they can construct well-defined cardinal concepts of small numbers—the conceptual basis for verbal subitizing. Verbal subitizing is a basis for meaningful learning of key number and counting concepts and skills in the progression (see Baroody, Lai, & Mix, 2006, for a detailed discussion).

- *Ordinal concept of small numbers*. Verbal subitizing enables children to see that "two is *more* than one" item and that "three is more than two" items, construct an ordinal (as well as cardinal) meaning of numbers, and understand what adults mean by the term "more."
- *Meaningful object counting*. Verbal subitizing enables children to understand the whys as well as the hows of object counting (instead of learning this enumeration skill by imitation or otherwise by rote). For example, by watching an adult count a small collection a child can recognize as "three," the child can construct the *cardinality principle of counting*: understand why the last number word in the count is emphasized or repeated—it represents the *total* or how many (the cardinal value of the collection).
- *Increasing magnitude principle and counting-based number comparisons*. Between 3.5 to 6 years of age, children's thinking about the counting numbers from one to ten undergoes critically important changes. By 4 years of age, children apply their ordinal concept of small numbers to their knowledge of the counting sequence. Specifically, they appreciate that the number words in the counting sequence, not only represent different, specific quantities but, quantities of increasingly larger size—the "increasing magnitude" principle (Sarnecka & Carey, 2008). Put differently, they learn that a number word further along in the counting sequence than another represents the larger collection (e.g., Schaeffer, Eggleston, & Scott, 1974). This enables children to determine the larger of two collections by counting them—the collection that requires the longer count is "more."
- *Number-after knowledge*. As children become familiar with the counting sequence, they no longer have to start with "one" and use a running start to determine the number after another (e.g., "After four is "one, two, three, four—*five*."). Instead, they access the counting sequence at the given number and state its successor. This skill is critical for the (efficient) application of the next two steps in the learning progression.
- *Mental comparisons of close/neighboring numbers.* The application of the increasing magnitude principle and number-after knowledge enables children to efficiently and mentally compare even close numbers, such as the larger of two neighboring numbers, without counting (e.g., "Which is more seven or eight?—eight").
- *Successor principle*. The increasing magnitude principle, number-after knowledge, and the mental ability to compare even neighboring numbers, however, does not necessarily entail a *linear representation of numerical magnitude—an understanding that each successive number increases by a constant amount* (e.g., that the difference between 8 and 9 is the same as that between 2 and 3). Such an understanding is embodied by a straight-line graph with a slope that is appreciably more than 0 (see, e.g., Frame A of Figure 1). (A straight line with a slope that is basically 0, as illustrated in Frame B of Figure 1, indicates viewing all numbers as having essentially the same magnitude.) Indeed, some research suggests that children begin with a *logarithmic representation of number* (e.g., Berteletti, Lucangeli, Piazza, Dehaene, & Zorzi,

2010; Siegler, Thompson, & Opfer, 2009). That is, their representation of the counting numbers seems to involve large magnitude differences between small numbers and increasingly smaller ones between progressively larger numbers. For example, the estimated difference between 8 and 9 is smaller than that between 2 and 3 (see Frame C of Figure 1). Young children may see "really large numbers" such as 8, 9, and 10 as almost indistinguishable in magnitude. In time, preschoolers use small number recognition to see that "two" is exactly one more than "one" item and that "three" is exactly one more than "two" items. By generalizing this induced pattern to the counting sequence as a whole, children construct the *successor principle*: Any number is exactly one more than its predecessor in the counting sequence. (The successor principle is a kindergarten goal in the *Common State Standards* or CCSS*,* 2010.)

- *Reconceptualization of the counting sequence as the (positive) integer sequence.* Theoretically, the successor principle is the conceptual basis and impetus for re-representing the counting sequence as the (positive) integer sequence: *n*, *n*+1, [*n*+1]+1, … (Sarnecka & Carey, 2008) and

***Figure 1*. Different representations of numbers (Ns) 1-10 (Δ = "difference"; ~ = "similar").**



in a linear manner. As Frame D in Figure 1 illustrates, knowledge of the successor principle should result in magnitude estimates that are not only *linear* in shape but have *a slope of 1*.

- *Informal mathematical induction*. Theoretically, the successor principle is also the developmental prerequisite for informal mathematical induction because such reasoning is not possible without a representation of the integer sequence.
- *Infinite succession principle (concept of infinity).* As the case of Nikki illustrates, informal mathematical induction can lead a child to conclude that the counting or natural numbers are unending or infinite (Baroody, 2005)—a concept that appears to be within the grasp of many, if not most, primary-level children (e.g., Hartnett & Gelman, 1998).

Issues

A number of issues need to be resolved. The first two involve developmental issues; the last three involve methodological issues that need to be resolved to answer the first two.

1. *How does informal mathematical induction develop? Is the proposed hypothetical learning progression accurate? What are the developmental prerequisites and when is it typically achieved?* Smith (2002) unsuccessfully attempted to identify developmental prerequisites of informal mathematical induction (see Baroody, 2005, for details). Normative data on when children typically develop this powerful type of reasoning is non-existent.

2. *What is the developmental relation between the successor principle and the number-after rule for adding 1: the sum of any number n and one more is the number after n in the counting sequence* (e.g., recognizing that the sum of 7+1 is the number after *seven* when we count: *"8"*; (e.g., Baroody, Eiland, Purpura, & Reid, 2012). The successor principle and the number-after rule for adding 1 are logically related but reverse operations. The former entails being given n and the n after and *determining that the relation between the two is +1*; the latter involves being given n and the relation +1 and determining that the outcome of $n+1$ is the number after n. Logically, the successor principle and mathematical induction may enable children to use concrete experiences with adding 1 to collections of 1 to 3 (e.g., seeing that two items and *one more item* is three items), known small n number-after relations ("three" immediately follows "two" in the counting sequence), and known large n number-after relations (e.g., "eight" follows "seven") to deduce the general number-after rule for adding 1. Psychologically, answering a successor-principle question entails solving a less familiar and relatively challenging missing-addend problem ($n + \square =$ the number-after n), whereas answering an add-1 problem involves solving a more familiar and relatively easy missing-sum problem ($n + 1 = \square$). Might the number-after rule for adding 1, then, develop before the successor principle or at least simultaneously with it?

3. *Does the definition of informal mathematical induction presented in the present paper on the "Formal versus Informal Mathematical Deduction" section make sense?* Should it be modified and, if so, how? For example, is there anything that should be added or subtracted from the definition presented in Footnote 2? Might informal mathematical induction be better characterized as developing in phases? For instance, in light of Smith's (2002) tentative results regarding modality, might an understanding that a conclusion is necessarily true in all applicable cases represent a relatively advanced level of proficiency with this type of reasoning?

4. *Are current tasks reasonable measures of informal mathematical induction and, if not, how might these tasks be improved?* The first step in the recurrence task used by Smith (2002; adapted from Inhelder and Piaget, 1963) involved gauging the initial equivalence of two transparent plastic containers ("pots"), which were either empty (equivalent initial state) *or* had one item (a "green cat") in one of the pots (non-equivalent initial state). A child was asked Question 1 (Q1): "*Is there the same in each, or is there more in one than the other?*" The second step involved actual additions to the pots and an observable outcome. After the child put three to eight green cats in one pot one at a time and simultaneously did the same with orange cats and the other pot, s/he was again asked Q1 and then Question 2 (Q2): "*Does there have to be the same in each, or not?*" The third step entailed actual additions but unobservable outcomes (i.e., identical to the second step, except that the child could not view the plastic containers and their contents. The fourth step involved hypothetical additions. The three key hypotheticals were (a) "*pretend to add six to each pot at the same time*"; (b) "*pretend to add a great number to each pot at the same time*"; and (c) "*pretend to add any number at all to one pot and the same number to the other.*" The first hypothetical was followed by Q1 and Q2 served as a warm-up. The second and third hypothetical served as the penultimate and ultimate directives and were followed by Question 3 (Q3): "*If you add [a great number/any number] here [pointing to one pot] and the same number to that [pointing to the other pot], would there be the same in each, or would there be more in one than the other?*" Might Rips, Bloomfield, and Asmuth (2008) be correct that this task actually bears on universal generalization (empirical induction), not mathematical induction? Put differently, is it mathematical induction if a child correctly answers Q3 as a result of using a pattern induced previously in response to Q1 and Q2? What new knowledge has the child generated as a result of using a discovered pattern AND reasoning logically—knowledge

that can be applied at a later date, or to a related but somewhat novel task, without the need for preliminary questions?

Might the "largest number task" (the series of simple questions asked of Nikki) be adequate to gauge informal mathematical induction or does the task need to be expanded (cf. Hartnett & Gelman's, 1998, extensive protocol of questions)? Might the "counting-pattern task" described next be a good candidate for an informal mathematical induction measure for young children? Define an *even number* informally in terms of a collection that can be shared fairly between two people with nothing left over and an *odd number* as a collection that leaves one leftover item after fairly sharing as many items as possible. Have the child determine whether each of two successive numbers such as 4 and 5 is even and odd. Have the child use a fair sharing (divvying-up) strategy to answer and record the results on a number list (e.g., green highlight for the 4 square to indicate even; red highlight for the 5 square to indicate odd). Then ask if 8 or 9 are each even or odd. If needed, repeat this sharing process and feedback and proceed with the next example (e.g., 13 and 14). The dependent measure is whether a child will realize that for any pair of successive numbers one must be even and the other must be odd.

5. *Are there additional tasks that could or should be used with primary-level children to gauge whether they can learn or use informal mathematical induction?* In addition to the few examples of informal mathematical induction cited in the present paper, are there other examples of such reasoning for which 5 to 8 year olds might reasonably be expected to be successful?

Conclusions

In conclusion, although children as young as five seem capable of acquiring the important thinking and learning skill of informal mathematical induction, it is not mentioned in current reform documents—despite the recent emphasis on mathematical reasoning (CCSS, 2011; National Council of Teachers of Mathematics, 2000, 2006; National Mathematics Advisory Panel, 2008). Clearly, though, research is needed to evaluate the proposed learning progression and other issues raised in this paper. This knowledge then needs to be translated into curricula and teacher guidelines so that such reasoning can be encouraged in a developmentally appropriate manner.

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