SOME PEDAGOGICAL AND PSYCHOLOGICAL ASPECTS OF MATHEMATICAL MODELING IN SCHOOL MATHEMATICS

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Abstract
What kinds of experiences might school mathematics students have that could lead to knowledge of mathematical modeling? How might they experience a contextual situation and a problematic approach, purposed to guide them to build-up mathematical models as abstracted re-presentations, and then use those to verify and extend their understandings of the problem context? How might a teacher stimulate and guide precollege students toward such educational goals and experiences? How might a researcher describe, analyze, and interpret such pedagogical and psychological elements as lived/living mathematical experiences? In this paper, I seek my answers to these questions, first by telling a “teacher’s mathematics classroom story” written as a composite retelling from my memories of dozens of my actual lived lessons with the same “view tube” exploration. Then, I offer a metaphorical paradigm for discussing “states of being” within experiential “flow.” Finally, I discuss four “layers of meanings” I can identify and infer from an analysis and interpretation of my “view tube” experiences.

Overview
We live in an increasingly technical global society in which complexity has become more apparent and more accepted in all domains of human endeavor. Fortunately, but predictably, our human capacities to perceive and accept complexity also appears to be increasing, along with our concomitant creation of technologically supported tools with which we seek to address complex problems.

Education, as a socially determined function, must aim to prepare students in adulthood to work on the vast variety of problematic situations in our modern world. It can be argued that it is critically important that today’s school curricula and teaching approaches should foster the construction of student knowledge and proficiency to mirror the adaptive and generative capacities needed to continue to meet these increasingly complex and globally significant problematic situations and developments.

Given the profound and pervasive role of mathematics in such complex problem solving, a sound and generatively empowering mathematical education for all students at all levels, is critical to fulfilling the global needs of the future. One dimension of this mathematical role is the development and use of mathematical models and modeling. It is the purpose of this paper to address that particular role as it might appear in school mathematics, as a critical component of a sound mathematical education for citizen living and specialist problem solving in this 21st century.

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My approach will be to offer one of the many rich problematic situations I have used with students and teachers across my life as a mathematics teacher and teacher educator. I will first present a hypothetical “story” of a classroom episode that is firmly grounded in my numerous varied lived teaching experiences with the “view tube” situation. I will then provide a reflective analysis to highlight both pedagogical and psychological aspects that I find from my art and experiential practices as a teacher. Finally, I will try to address needed construct analysis and research that would aim to deepen our understandings of model building and modeling in order to improve school mathematical education.

**What do you see? (Orienting, Exploring and Experimenting) [Day 1]**

The central problem of an education based upon experience is to select the kind of present experience that lives fruitfully and creatively in subsequent experiences.

—John Dewey

“As you can see, I’ve given each of you a paper tube,” I said as I sought to get my typically energetic group of grade eleven students quieted to begin our lesson. “I’ve seen many of you already looking through your tube. Who might help us start by suggesting something they’ve observed when looking through their tube?” Quickly, several students raised their hand, ready to offer an idea.

“Juanita?” I asked of one of my more quiet students to whom I’d give a toilet paper cylinder.

She responded in a typically quiet voice, “Well, I’m not sure what you want, but it was round, no matter what I looked at.”

“A good observation,” I replied. “Michael, what did you see through your tube?” I asked of one to whom I’d given a much longer tube from gift-wrapping paper.

“Mine was small, really tiny,” he offered, as some of the students tittered, apparently ready to tease Michael about the way he said “tiny.”

“So, how do you think Michael’s view might compare to Juanita’s view? How might they differ? I asked. “Which might be the larger image, and why?” After several volunteered, I selected Pete, whose tube from a kitchen paper towel roll had a length between the other two.

“Well, I think what Michael sees is so small because his eye is a lot farther from the open end,” Pete offered.

“Very good idea, Pete. But, hold up your tube, and tell me: How do you think your view would compare with the other two?” I asked. Pete looked through his tube, but only shrugged and didn’t offer a reply. “Who might have an idea about my question? Jamie, what do you think?”

“Well, because Pete’s tube is between the other two, I think the size of his would be in between,” offered Jamie.

“Yes, that would be a reasonable suggestion; good reasoning. But, do you think it would matter what they’d be looking at? Let me ask you to think about this: Each of you, look through your tube. Now, quietly, look at three different places in the room. Look at something close to you, then something farther away, and finally something even farther from you. Here’s my question: how does the distance to what you are looking at, seem to affect what you can see?” I asked.

In a few minutes, most of them had ideas to offer. After we listened to several students report, I asked the key, starting question. “Class, here is what we are going to investigate today.

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1 Taken from Hatfield, L. Stories from My Teaching: Experiences in Mathematical Thinking (prepublication draft).
What is the relationship between the distance of our eye from the view tube object, and the size of the image of that object?”

In the next few minutes, using what I’d set-up (two vertical meter sticks taped collinear on a wall, and masking tape strips on the floor along a straight path one meter distances from that wall), I had students demonstrate while I explained the five similar paths I’d prepared in the cafeteria, and what each team would be doing to collect and record on the data chart I had given them—the measures of the image seen on the two meter sticks taped to the wall, with the observer standing at successive meter distances from the wall and looking through their tube.

In terms of potential sources of error in their data, we identified and discussed the importance of having the viewer’s eye directly above the edge of the masking tape I’d put on the floor for each meter distance, and to avoid tilting one’s head but look along a line horizontal to the floor. After discussing the use of repeated measures to help address potential errors, each team was given two rolls (toilet paper and paper towel, having very similar diameters); each data chart provided spaces to record measures for each roll, taken when two members of each team used each tube. Teams were named; just before we left for the cafeteria, I asked each team to write on the back of their data sheet their conjectures about what we would find.

In the next half hour, I observed, advised, guided, clarified and corrected the teams as they busily worked to perform the view tube experiment and collect their data. Each student had a role, including using two colored paper strips to hold against the meter sticks to mark, as the viewer directed, where the top and the bottom of the image was seen. Before finishing, all students copied the team data onto their own data sheet.

Back in the classroom, as the class session was ending, I collected all of the team data sheets, and directed the students to prepare for the next class by using their personal data sheet to study the data in search of ideas about the behavior of the view tubes.

**Collecting and Using the View Tube Measures (Data Analysis) [Day 2]**

We don’t learn from experience; we learn from reflecting on experience.

~ John Dewey

“I hope you each were able to review your team’s view tube data, so you have ideas to suggest now. Please note that I have entered all of our data into the spreadsheet you see on the screen, so as we talk we can refer to that,” I said as we began class the next day. “Who can start our discussion with an idea about how the view tubes behave?”

Billy, who was one to raise his hand often, stated after my nod to him, “I could see from our numbers that the size got bigger.”

“Very good, Billy. Now, can you try to say a bit more? The size of what got bigger, and when did it get bigger?” I asked.

“Well, up there our data is shown as Team C,” he said, pointing to the chart on the screen, making this identification by matching what he had on his sheet with what I’d put into the spreadsheet. “And you can see, as our viewers moved away from the wall, the values from the meter sticks get bigger.”

“Does everyone agree? I asked the class; many nodded agreement. “Does this happen for each team, and for each of the tubes?” Again, all seemed to agree. “Excellent. So, we can say that as distance from the wall increases, the image size increases, right?”

At this point, Meredith raised her hand, so I nodded to encourage her to speak.

“OK, I remembered how we’ve always made graphs from data, so last night I made graphs from our data,” she offered, as she held up her sheet of graph paper. I could quickly see she’d plotted points for the two data sets, using the same axes.
“Please come up front, Meredith, and hold up your graph for all to see,” I said. With a bit of shyness, she did this. “Did anyone else try this?” I asked. Only a few others raised their hands. “OK, now I could ask each of you to do this, but in fact I expected that we’d want to see the picture of our data, so I’ve already done that on the spreadsheet! Remember, computers are tools to help us, and as we know, Excel will produce data graphs for us. Let me show you what each team’s graph looks like.”

I paged through the tabs, and paused enough to identify each team, and ask one student from the team to comment. Most said something about the points appearing to “be a line” or “to be linear,” and on each graph there were two linear patterns of discrete dots, one for each data set from each view tube. When we got to Team D, we found one of the points did not seem to “fit” on the line, and I asked, “OK, team, what do we think is going on with that point?”

A vigorous exchange occurred among the students, but it was suggested that somehow they had “messed up” their measuring or their calculation or their recording of that data value. This led to some comments about how important it was to have more than one data set.

After we’d reviewed and discussed all of the graphs, I suggested, “Remember, these graphs were produced using the averages of measures found from two people viewing. Sometimes when investigations include multiple trials that produce data gathered under identical conditions or constraints, it is OK and useful to use averages across all of the data. What might be some of the advantages for doing that?” I asked.

Several suggestions were voiced, all pointing at the usefulness of repeated measures helping to avoid errors, and perhaps producing a more accurate idea. “OK, I’ve made one more graph for us to see, using all of your data to find the average image size for each tube at each of the distances from the wall. Before I show you that, what do you expect we’ll see?” I asked.

Most replies predicted that the points would appear even more like straight lines (collinear). After I displayed the composite graph (see Figure 1), students expressed positive assurances that they were right, and that it made sense.

![Viewtube Experiment](image)

**Figure 1.** Average Image Sizes for Two View Tubes (Average Diameter=4.25 cm)
“So, let’s dig a little deeper, using these last graphs. Notice, these are not connected points. Why should these be left as discrete points? Jason?” I asked.

“Well, we really shouldn’t connect them, because in our experiment we didn’t look at any distances between each meter, so technically we don’t really know anything about the size we’d see in between,” he explained.

“Very nice explanation, Jason. Anyone want to add to that? OK then, here is another question. We see that the graphed data points from each view tube appear to lie on different lines. What can we say about those lines?” Hands were raised. “Try to use algebraic ideas we have learned about graphed lines. Juanita, what do you see?” I asked.

“Those two lines have different slopes, but they would intersect, and that would be at the origin,” she asserted, exhibiting more than her typical level of confidence.

“Wow, that is a great conjecture, Juanita!” I exclaimed with enthusiasm. “But, can we think about that a bit more? Think about the eye; how close can the eye get to the wall?” I asked. After some discussion, I asked Juanita to model with each tube, to show that her eye would have to stop at a distance from the wall equal to the tube length. “And, in each case, what would be the size of that image?” Quickly, it became clear that it would be the size of the tube diameter, and since these were equal for the toilet paper (TP) and paper towel (PT) tubes, this would be the value on the y-axis for both tubes. And, after further discussion, it became clear that the smallest “distance” value had to be the tube length, so the left-most points would be (11.4, 4.25) and (28.9, 4.25).

“So, we see that the lines would not intersect all. But, I wonder, could we say something more about their slopes? Recall, slopes are numbers, right? And the values of those numbers can tell us something about the ways the two variables are related. So, let’s be reminded of what the two quantities are here. Steve, can you identify the two quantities?”

He hesitated, and I thought he was not going to reply, though he’d been vigorously waving his hand at my question. Finally, he said, “Well, I’m not sure how to say it, but the steeper one has the bigger slope value. And, that is for the data from the shorter tube. So... the longer the tube, the slower the image size grows.”

I saw several frowns and puzzled looks, so I asked, “Steve, I think you have said it correctly, but I’m wondering if you can repeat it for everyone? Perhaps to help, I’ll label the two lines.” I quickly made text boxes, and typed “TP” and “PT” next to the plotted points. Using the computer pointer, I said, “These are the plots from the shorter toilet paper, or TP, roll, and these are from the longer paper towel, or PT, tube.”

With that, Steve said, “OK, the paper towel line has a smaller slope, so that means that as the distance from the wall increases, the image size grows, but not as fast as for TP.” This time, most of the students nodded their agreement.

“Well done, Steve. Alright, next I want you to get back into your teams, and see if you can estimate those slopes, and then see if you can write the equation for each of these lines. I’ve printed the Excel averaged data and plots of the two data sets we’ve been studying, so here is a copy for each you to use,” I said, as I distributed the printout to them.

In the next few minutes, the teams discussed ideas, and worked to produce their results. I concluded with their challenge for tomorrow: “On the back of this last sheet, tonight I want you to try to produce a geometric sketch that will represent our experiment. Let’s agree: you don’t have to make a realistic drawing; rather, try to reason the basic geometry of what we did. Try to make a two-dimensional representation of the essential lines and shapes and distances and sizes. Here are a couple of clues: if we put it into the flat plane, what shape will be good enough to represent the tube? It won’t be a cylinder. Think of how it would look in a side view of the scene.” After working together in their team groups, the class ended and the students departed.
Visualizing, Re-presenting and Model Building (Geometric Analysis) [Day 3]

Problems cannot be solved at the same level of awareness that created them.
~Albert Einstein

As my students entered the classroom the next day, several stopped to show me the drawings they’d sketched, and I could see a variety of representations. As they quieted, I asked Deanna to come to the front of the class. “Here is one of our view tubes, Deanna. Please turn to face the left sidewall, hold the tube to your eye, and look through it. Now, class, look at what Deanna is showing you. What is the geometry of the side profile that you see? Then, look at the sketch that you have made for today. Is your drawing a good geometric representation of what Deanna is modeling for us?”

Many of them nodded, but a few of them began to make a new sketch, or erase to revise what they’d brought with them.

“So, Deanna, I want you to keep showing us, while I’m going to try to make a drawing here on the whiteboard, with help from all of us, and my meter stick. OK, how might I show the sidewall,” I asked.

With various students guiding, I used the straight edge to place a vertical (left side) and a horizontal (below) perpendicular segment for the sidewall and the floor. Next, they helped me to position and draw a rectangle to represent the side view of a tube, and we noted how its sides were perpendicular and horizontal to the prior segments. We agreed we didn’t need to show a person, but I did mark the midpoint E of the rectangle’s “back” (right) side to show where the eye would best be located (and I added an artistic side view of an eyeball with eyelashes; this caused some tittering)—just as Deanna was modeling when she looked through her tube to the left sidewall. From Mark’s suggestion, I also drew a dashed line segment from the “eye” dot, E, perpendicular to the “wall” segment.

“That segment shows us the distances we recorded,” added Mark.

“Well, Mark, recall the distances were marked on the floor, but if the eye was directly above, then we can use the eye-to-wall distances, too,” I expressed with excitement. “So, here we have shown a model of the experiment, and we can see that the measure of this eye-to-wall segment would be one of the quantities that we varied,” I stated as I pointed.

“Now, here is a very key step in our geometric analysis of this representation. Imagine how the ‘eye’ would see through the tube to the wall. Can you tell on the wall where the exact points at the top, and at the bottom, of the image would be located?” I asked as I used the meter stick as a pointer. Several students immediately raised their hands, but I added, “Describe by naming points how the line-of-sight would find the top point of the image. McKenzie, will you please come up here, and show and tell us what you think?” I asked: I’d noted before that she often doodled with quite artistic designs, possibly signaling her visualization abilities might be stronger than others.

I handed her the meter stick, and she used it to point at E, slowly sliding it inside the rectangle to the top left rectangle vertex (U) and continuing to stop when she pointed at a spot on the vertical wall segment. "And, can you show how you’d locate the bottom point of the image?" I asked. With a pointing motion, she started again at E, moved to the lower left rectangle vertex (L) and stopped at some point on the 'wall' segment. "Very well done! McKenzie, could you place the meter stick against the white board to show us more precisely what you just demonstrated?" I asked. She was able to do this, and I drew the two line-of-sight segments, labeling the endpoints for the image as points T (top) and B (bottom), and adding “i” as the expressed variable measure for length of the image.
“OK, class, lets review what we have here,” I said, as I began to transition to their next activity. In the next few minutes, I reviewed all of the key geometric relationships in the drawing we’d constructed. “Now, listen closely. You will next work in your assigned pairs on our computers with GSP, and your task will be to start to construct what we have shown here in our whiteboard drawing. Before we do that, are there any questions about this representation?” I answered a couple of questions, but most students seemed eager and ready to get started.

“One last thing—I want you to do this construction on a graphing window, so first you will open that, and since we will only need the first quadrant, you can drag the origin to the lower left corner of the window. Where do you think we’ll place the wall segment, Billy?” With a shrug, he offered it would be the y-axis. On the whiteboard, I then showed a y-axis, collinear with segment TB.

“So, would you agree that we can use the x-axis to represent the floor in our experiment?” I asked; all nodded agreement. “Good. Now, here is a very key idea for your GSP construction. Because we want the distance from the eye to the wall to vary, we are going to determine the point E as a variable point. Who might have a suggestion for how to do that?” I asked. A variety of ideas were voiced, but none were clear.

Finally, I offered, “OK, from our experimental data, we’ve seen how the distance from the wall determines the size of the image. Do we remember from our discussions, that for a function, the values for the dependent variable are shown with values from the y-axis while the independent variable is found using values from the x-axis?” I asked. Most nodded their agreement, but Scott asked, “I’m still not sure how to do this.”

Using the white board drawing, I quickly demonstrated the first key steps in GSP for constructing what we’d shown. In the remaining class time, all pairs were busy at their computers, developing the initial GSP construction setup. I helped at key places, so all were able to construct workable, but incomplete, representation. As the class period was ending, I directed them to save their file, so they could continue in the next class discussion.

\[\text{Figure 2. Construction of a GSP Model of the View Tube Situation}\]
Modeling, Analyzing and Deriving (Algebraic Analysis) [Day 4]

If you can’t explain it simply, you don’t understand it well enough.
~Albert Einstein

The next day, the students immediately re-opened the file on their computers they’d saved at the end of the prior class session. I had found, and placed one of the student files on my computer, and had it opened on the digital projection. After about fifteen minutes, when it appeared that most of the teams had arrived at a critical point in the construction, I interrupted them to ask them to look at the projected display.

“It time for us to discuss how we are going to get the model to generate and handle the data. First, let’s see how we can model the independent variable so it portrays what we know and understand about it. How can we show the quantity that is the distance of the eye to the wall, so that we can model it as a variable? Look again at the drawing we made yesterday on the white board; I’m asking about the measure, ‘d,’ and how we might make that vary in the GSP construction. Does anybody have an idea?” I asked.

The students looked to each other, but there did not appear to be anyone who had a suggestion. Just before I was going to explain, Meredith hesitatingly raised her hand. “Ah, Meredith, what’s your idea? I asked.

“Well, I’m not sure about this, but I was remembering how in algebra we made a sliding point on the x-axis, and when we moved it the values of ‘x’ changed. So, here the value of ‘d’ is varying as the distance from the origin O to X, so can’t we just use that measure?” she asked.

“Did everyone hear what she just said?” I asked. “Meredith, I think you have a great idea, so will you please say it again, and this time, will you all listen carefully? And, please watch me, as I use the drawing on the whiteboard to model her idea,” I directed.

She stated her ideas again, and I marked and pointed on the drawing to demonstrate it. Then, I demonstrated with the displayed GSP window how we could measure OX, which would vary as the distance of the eye to the wall changed (point E is moved).

“Now, what is the other quantity that varied in our experiment?” I asked. Many replied that it was the image size. “How can I display the measure of that quantity? James?”

“Well, on your GSP, you need to show those line-of-sight segments we talked about yesterday. And, where those intersect the wall segment at T and B, find that length.”

Very good, class. OK, let’s stop so each of you can add these last ideas to your construction before we continue,” I said. As always, while they worked I circulated to ask and answer questions---my way of assessing understandings while making sure each had produced a workable representation.

“It looks like we each now have a geometric model of the key ideas from the view tube experiment. Please look up at the display up here, and check each other’s construction to see if it fits with what we have here. And, be sure to try your movable points: X for the location of the eye, L to vary the length of your tube, and W to vary its diameter,” I directed. After a few questions to clarify for individual differences, I asked, “Now, you should have at least two measures showing: one for the distance of the eye from the wall, and one for the size of the image on the wall. Does anyone have an idea for how we could plot those measures as an ordered pair, and what would be the order we’d want in that pair?”

Immediately, several hands were raised. “OK, Jennifer, what would you say?”

“T see what we are going to do!?” she said with excitement in her voice. “When we plot that point, it will be like we plotted our measured data! So, I think the order will be the eye distance OX for the x-value, and then the image size TB for the y-value, because that is how we graphed them,” she asserted confidently.
“Who agrees?” I checked; almost all nodded or raised a hand. “OK, on your GSP window, can you select the values in that order Jennifer said: first the distance measure, then the image measure, and then in the ‘Graph’ menu, select the ordered pair choice?” I emphasized. “When GSP shows the plotted point, check it to be sure it fits with the measures you have on your screen. Did you plot the correct point?” I asked. Quickly, I added, “Now, while the point is still showing selected, set the ‘Trace point’ option in the ‘Display’ menu.”

While I was waiting for all of them to get those steps completed, I circulated to check on their actions. Matt raised his hand, and began excitedly to wave it. When I got to him, he proclaimed, “I tried moving the X point, and it plotted lots of points, and they are making a line!”

“Ah, very good, Matt! Would you please complete these last steps on our GSP display, and show everyone what you’ve seen?” While he was doing that, many others had overheard and produced similar results on their display. When finished, Matt demonstrated and explained. Jeremy asked Matt, “How come your line looks steeper than mine?” By reviewing the data graphs they’d produced earlier, it was quickly suggested that the greater line slope might result from shorter tubes. This set the stage for the next phase of activity.

With my guidance, the students explored the use of the GSP model to produce graphs of the simulated data for the view tube experiment. First, I guided them to select and measure the segments, tube length and tube diameter. Each placed X at the origin and then using movable points L and W we set the tube length (4 cm) and diameter (1 cm); they agreed this looked similar to the toilet paper roll. By moving X slowly, they produced a linear graph which I labeled “l=1;” I had them move the tube back so X was again at zero. “OK, without erasing that trace, now move point L to double the length (8 cm). Before moving X, what will you predict about the graph?” I asked. Shortly, most appeared to be satisfied with the results. “Now, later class we’ll come to understand very clearly how tube length affects the image size, and specifically determines the slope of the linear relationship.”

![Figure 3](image.png)

*Figure 3.* Comparing (distance, size) Results for Various Tube Lengths (1 to 10 cm)
Next, I guided them to use the model to explore to find how a varying tube diameter would affect the image size, when the eye does not move when set at (25, 0) so the distance is constant and the tube length is also fixed (constant). Before starting, I asked for their predictions; only a few had ideas, and most guessed it would be a line. Indeed, for each length, as the diameter varied the plotted points showed a linear relationship; as they systematically set longer tubes, the slopes decreased. This led to some good explanations for why that was happening. Again, I assured them that soon we’d see clearly how those slopes were determined.

Finally, I asked, “What other set of values might we explore that could affect the image size?” Several suggested that we could set the distance (leave X on the point (25, 0)), fix the diameter (1 cm), and vary the tube length; again, I asked them to predict what they’d see, and many again guessed it would be a line. Quickly they tried this, and numerous exclamations of “Huh?” and “Wow!” and “What is that?” and “Hey, it’s a curve!” could be heard. Many appeared to find pleasure in being surprised; others expressed curiosity about the curve. Several got the idea they could produce differing curves by changing the tube diameter.

After further explorations by them, I asked, “Now, shall we see more clearly how each of these relationships behave? We’ve seen two different families of lines from the pairs (distance, image) and (diameter, image), and a new kind of curve for (length, image). The key to understanding is to return to our basic geometry,” as I directed attention back to the white board drawing.

![Diagram](image)

**Figure 4.** Sample GSP Results for One “Length” Experiment
With my questioning, they quickly identified the triangle similarity that allowed us to express the key proportional sentence:

tube length : eye distance = tube diameter : image size.

They were quickly able to write the “cross product” sentence:

\[(\text{length})(\text{size}) = (\text{diameter})(\text{distance}).\]

But, when I asked them to write the algebraic equations that would express the three functions for image size depending on eye distance, on tube diameter, and on tube length, many of them became confused and needed my assistance. Yet, with their input, I was eventually able to write on the white board:

1. \(\text{Image size} = \frac{[\text{diameter} \div \text{length}]}{\text{(distance)}}\) \(\text{image} = f(\text{distance}).\)
2. \(\text{Image size} = \frac{[\text{distance} \div \text{length}]}{\text{(diameter)}}\) \(\text{image} = f(\text{diameter}).\)
3. \(\text{Image size} = \frac{[\text{distance} \times \text{diameter}]}{\text{(length)}}\) \(\text{Image} = f(\text{length}).\)

From these, students saw how the quotients of fixed measures (diameter/length) in (1) and (2), determined the slopes of the lines. They looked back at their first data to see that their measured values produced a good approximation of the slopes for their graphed lines, and also for the plots of averages (4.25/11.4=0.373 and 4.25/28.9= 0.147).

Finally, I used their surprising curves as a way of introducing the idea of indirect variation, the graph of the hyperbola, and our study of that conic section.

**Experiential Analyses and Interpretations: Pedagogy and Psychology**

In keeping with a traditional approach to didactics (Balacheff, 2009; Brousseau, 2010; Freudenthal, 1983), and using my lens of lived/living mathematical experience (Glasersfeld, 2007; Maturana, 1987; van Manen, 1990), I offer the following analysis and interpretation. My didactical assumptions include that a teacher’s pedagogical theories, perspectives and practices are formed and function in seamless interplay with one’s knowledge and beliefs related to mathematics, intended curriculum, and a psychology-sociology of oneself and one’s students experiencing classroom mathematics. In this sense, prior knowledge, impacts of lived experiences, and perception-in-the-moment fuel enacted teaching approaches, decision-making and actions, and in turn the real time (living) classroom experiences of the teacher become a dominant influencing feedback factor on the pedagogical choices and decisions of the teacher. As researchers, we strive to witness the living, enacted pedagogy. Across a teacher’s experience, these reciprocal factors within a reflectively based practice can become influencing, anchoring guides leading to a teacher reinterpreting and altering her theories and subsequent practices. Such constructions and reconstructions of personal pedagogical theory appear to relate to one incisive concept from Glasersfeld’s (2007) radical constructivism:

What we ordinarily call ‘reality’ is the reality of relatively durable perceptual and conceptual structures which we manage to establish, use, and maintain in the flow of our actual experience. This experiential reality; no matter what epistemology we want to adopt, does not come to us in one piece. We build it up bit by bit in a succession of steps that, in retrospect, seem to form a succession of levels. (p. 36).
Metaphor
What I see as classroom mathematical ideas and actions arise interactively and naturalistically within lived mathematical experiences of both a teacher and her/his students, and these occur in ways that mirror the inherent flow of possible mathematical developments that honor, yet are responsive to, their perceptions of the nature of the posed situations (contexts) and one or more possible logical (rational) constructions of those. Thus, I will develop my analyses and interpretations within a concept of lived/living mathematical experiences built upon a metaphor of flow (Csikszentmihalyi, 1990).

My primary metaphor is that experience (including joint classroom experiences) is like a living, flowing stream of water, and with respect to my hypothetical stream, I (and others I perceive in relation to my stream) may exist in various states of being (Hatfield, 1991). In attempting to describe, analyze and interpret such states of (individual but shared co-) existence, a constructivist must accept the ultimate failure to know those, except in second hand, self-constructed ways. As such, a constructivist observer-researcher, as a source of the descriptive data, analysis and interpretation of experiential being, necessarily will also be a primary source of potential idiosyncratic bias and error in that data.

“States of being” within the metaphor of “stream flow” may have many, complex elements and dimensions of interpretation. Consider these possible “being” relationships:

- Being separate observer (SO), such as on the stream bank --- not “in” but only “seeing” the flow of experiences; real presence versus viewing video replay differ in powerful sensory ways.

- Being involved observer (IO), such as standing still in the stream --- sensing awareness of some elements or aspects of local flow moving past, yet not moving “in” or with it;

- Being participant/observer (p/O), such as floating on top the stream --- again, sensing many elements of the flow, yet being carried along in the flow and still inherently apart from many complex sensory interactions vital to a more full awareness;

- Being participant/observer (P/o), such as being carried along more fully “in” the stream flow (standing/floating/immersing) --- sensing/perceiving/interacting much more directly connected to and aware of the essences of the stream itself;

- Being “it” (BE) --- projecting to enter the interiorized existence, to “be” the living stream, sensing directly the global, holistic richness and also many local complexities of experiential flow, such as shape factors of shoreline, stream bottom, and objects in and on the stream affect the nature and quality (essences) of the existential, phenomenal flow of “being.”

In these differentiations, I am suggesting that the roles of teacher may provide a very powerful (yet still necessarily limited) vantage point for engaging in efforts to understand lived classroom mathematical experiences as states of being embedded within an experiential flow of individuals and a group that includes oneself. In this, students and teacher will each BE in their experiential streams, but through the dynamics of social sharing these streams interact, co-mingle, and even in some sense merge. Moreover, a teacher who has repeated (yet surely highly varied) experiences across many lessons involving similar intended goals, activities, and student interactions will better be able to build a cumulative, collective understanding of the nature of the dynamics of such intended and interpreted experiences. In this belief, I seek to offer what I now understand about the nature and quality (essences) of apparent possible mathematical experiences of students and teacher within my instruction using the view tube situation.
Analysis and Interpretation

Like probing layers of an onion, I now see many layers of potential meaning expressed in the foregoing composite portrayal (“story”) of my repeated experiencing of the view tube investigation. And, for me, these meanings embody many important aspects of my pedagogical and psychological analysis of mathematical model building and modeling as it might occur in school mathematics.

**Layer 1.** As a first layer of meaning, I can identify overall experiential elements in the progression of lessons ---purposes, strategies, experiences, and outcomes (for both teacher and student). These more global, comprehensive, and action-oriented aspects include:

- Posing/perceiving the view tube situation --- gaining student attention, motivating/building interest, asking/answering questions, developing/extending involvement, framing/clarifying approaches, setting up/tasking the investigation;
- Conducting the investigation --- identifying quantities and measures to be made, specifying independent, dependent, and controlled variables, clarifying data procedures, finding and controlling sources of error, guiding data collection;
- Exploring to understand the data --- organizing the data, identifying outlying values, looking for trends or patterns, noting how data varies, graphing the data, analyzing the graphs;
- Conjecturing data-based relationships --- interpreting data to frame a relationship, checking the possible relationship with the data, expressing it mathematically, applying the relationship to predict;
- Analyzing situational geometry toward a re-presentation and theory --- reviewing the enacted situation using geometric ideas, re-interpreting with 2-d geometric ideas, developing analytical steps to build up a 2-d sketch using abstracted representations of key elements, developing student comprehension and acceptance of the re-presentation-al variables and implied validity;
- Translating geometric relationships to build a dynamic graphic model --- re-interpreting sketching steps and discussion into the GSP constructional actions, making and explaining movable points key to a dynamic model, measuring key variables, using and interpreting measures to plot varying points;
- Testing to confirm the computer graphic model --- producing and checking linear graphs for image size depending on eye distance, comparing to experimental data and lines, exploring to contrast slopes of lines for differing tube lengths;
- Using computer graphic model to investigate and conjecture new relationships --- shifting to question image size depending on tube diameter, producing and interpreting new linear graphs, framing new relational conjectures, studying image size depending on tube length, producing and interpreting a new curved graph (perhaps to introduce the hyperbola);
- Re-analyzing the geometric representation to derive formal algebraic (functional) relationships --- returning to a deeper geometric analysis, identifying geometric similarity and resulting proportionality, expressing the four variables algebraically, interpreting three functional expressions to determine image size; review to confirm prior results and explain effects of two constant values.
In many ways, this first layer of meaning could be realized by what I called previously an “observer” (separate SO or involved IO)---metaphorically, as one standing on the bank or “in” the stream, witnessing the passing events, noting conceptual and temporal developments, listening to verbal interactions, perceiving participant actions and re-actions, attending to dynamics, probably missing certain important aspects, but never directly engaged in the experiential flow, per se. Any educator who has experienced the dramatic differences to be found in observing someone else teach, versus your own teaching or observing oneself teaching via a video recording, knows the differences among an amazing array of textured details, meanings, and significances manifested in witnessed versus direct, firsthand experiences versus a reflective re-witnessing with analysis of such direct experiences. As such, these experiential elements appear to me as familiar pedagogical lexicons, yet also connote a rich variety of intellective and emotive psychological aspects for both teacher and student.

Layer 2. At a next layer of detailed meaning, I can identify these elements and experiences that include second order abstractions, as purposed and perhaps (apparently) experienced:

- Enacting deeper understandings --- These efforts and apparent results permeated much of what was intended and done. Through successive stages students were led to address the basic view tube situation in progressively more abstracted, meaning-making, and formalized ways. One assumption may be that the depth of experiential engagement/involvement by the student would directly affect her/his depth of understandings related to the situation and its “mathematization.”

- Reinforcing and strengthening phenomenal concepts and relationships --- The directly experienced view tube phenomena stimulated all developments, and yet remained vitally connected to the mathematical abstractions. Student “real world” experiences served as the generative source for proposed relationships, and later these continued to serve to link to the imagined “actions” implied by the model.

- Formulating and interpreting viable representations of reality and experience --- It is a truly “mathematical” act to use geometric ideas to re-present reality and experience, and to experience and reflectively recognize this step involves a powerful second order abstraction that can reveal the power of “mathematizing.”

- Deconstructing and reconstructing pictorial-to-graphical dynamic computer model --- Another level of abstraction was required of the students to move their thinking from the sketch (itself an abstraction of reality) to the construction of the GSP dynamic re-presentation, and then to interpret those variable representations as measured quantities to be graphed as a set of ordered pairs in order to portray (re-present, again) a potential functional relationship.

- Connecting a computer dynamic graphical model back to prior experiences --- Building the GSP model involves thinking “away” from their phenomenal, reality experiences, but then using the model to produce results aimed at thinking “back” to their prior experimental experiences provides powerful, reversible confirmatory reasoning—as a kind of positive feedback loop in thought.

- Applying generatively the computer model --- Realizing this level of thinking offers another depth of meaning, wherein students can see the power (and a purpose) of model building can lay in the eventual use of the model as an exploratory or investigative tool in order to generate new findings. In my numerous lessons I believe only a few students (and teachers) were typically having this realization.
• Deriving formalized re-presentations of functional relationships --- For many mathematics teachers, this may be the most important (perhaps only) purpose in using situations like the view tube. Because it is the most abstract and formal stage, my pedagogical view is that most students will benefit from the richness of meanings developed within an experimental, data based foundation, followed by a purposeful geometric analysis and application into GSP, prior to attempting the most formalized development. Further, developing the three functional expressions last provides students with a basis of explaining how the two constant quantities are used to explain slopes of lines and shapes of hyperbolas.

Such meanings could possibly be derived by some level of “participant/observer” (p/O or P/o) with the depth of penetration perhaps depending on the extent or quality of direct participation in the lived flow of experiences---a common remark, “you just had to be there!” My speculation is that some of these identified meanings could be formulated well by an observer-researcher who was also able to circulate and sometimes directly encounter students as they engaged in lesson activities, and encounter the teacher via expanded discussions before and after the lesson, wherein such enriched contacts with the flow of teacher and student experiences could lead to deeper observer meanings of the lived experiences.

Layer 3. Probing into another layer of experiential meaning for me, I can ponder the even deeper, broader mathematical and psychological impacts upon students and myself to include the following:

• Grasping holistically the view tube activities---conceptual and procedural understandings sufficient to interpret the “real world” view tube phenomena in order to formulate an experimental paradigm to produce intentional measurements aimed at exploring each variable and co-variations toward predictable relationships;

• Comprehending the “finished” mathematics of the view tube---operational, explanatory understandings of a formalized multi-variable theory for specifying view tube image sizes as functional dependencies involving tube length, diameter and eye distance;

• Realizing the view tube developments as an instance in theory building---reflectively enacting meta-understandings of the role and nature of the view tube problematic “lived” experiences as instantiations of strategic and tactical approaches aimed at building up an abstracted mathematical theory that can be both particularized and generalized;

• Recognizing generalized, structural aspects of the nature of an intellective schema for mathematical model building and modeling that leads to accommodations within a new level of operational quantitative reasoning;

• Appreciating with genuine affect the significance of the capacities of the human mind to engage in rational problem solving in ways that can lead to abstractions, generalizations, models, and theories originating in ordinary phenomenal situations.

To me, these impacts would occur from only the most “in”-volved experiences (P/o or nearly BE) from which deep levels of reflective meanings would lead to abstractions of highly generalized quality. Though not exemplified in my “story,” in many of my lived experiences with groups, and especially teachers, I was able to engage them in reflective discussions aimed at stimulating and guiding them toward such deeper realizations, and many were able to demonstrate elements of such deeper meanings and impacts.
Layer 4. Finally, at a most core layer of meanings, I would conjecture some anchor points for a teacher’s and a student’s cognition and affect connected to their lived experiences from such view tube episodes could include the following:

- Perceiving, attending and engaging --- The nature and quality of the lived experiences of students and myself in view tube scenarios must surely be deeply dependent upon how each of us were involved in the shared events. As a teacher, I’m deeply attuned to my roles in establishing and maintaining student attention and engagement, and intentionally employ pedagogical moves to stimulate motivation, to foster each student’s involvement, to sustain high levels of student activity, and to react in specific ways when my experiences with them lead me to perceive their behavior indicates less engagement than what I expect or want. Thus, within our shared experiences psychological factors (Piaget, 1975) related to reciprocal cycles of perception, attention and action dynamically occur to propel us within the experiential flow (Glaser, 2007).

- Persisting to solve --- One important interior factor that may function to support the direct involvements essential to higher quality experiences is the participant’s sustaining will to “try, and to keep trying.” My pedagogical approach to the view tube development may be characterized as “problematic” in nature---I strive to emphasize posing and asking and solving toward feasible developmental stages, rather than telling or showing (Polya, 1973). One characterization of ensuing experiences might be the existence within the flow of a sequence of “local guided problem-solving” episodes. A detailed analysis of these would surely reveal multiple purposes, results and “living” experiences, but where “making sense” is a central “problem to be solved” by the student; indeed, that may be the most fundamental, pervasive type of classroom “problem solving” experienced by individual students. Within this “problematic approach,” sustaining persistence becomes critically important, and we could ponder or explore the nature of the lived experiences with respect to this psychological and pedagogical factor.

- Abstracting and generalizing --- For me, these are core, pervasive, adaptive human cognitive processes, that develop within and across essentially every context of one’s living and life (Glaserfeld, 1991). Yet, we see mathematics as a context in which these processes and constructed results are dominantly essences of “knowing mathematics.” As such, the flow of experiences possible in the view tube episodes must surely be anchored for both students and teacher in a psychology and pedagogy that vitally draws upon, and impacts upon, the individual’s states of being for engaging in abstracting and generalizing (and reciprocal processes of exemplifying and particularizing), with constructions we might call abstractions and generalizations (as identifiable “items” of knowledge).

- Reasoning --- For me, this represents a most fundamental element in what is meant by “doing mathematics,” and therefore the essence of what I mean by a “mathematical experience.” It is a belief that the substance (content) of what is called mathematics, approached without an emphasis upon human mental reasoning, mostly likely fails to engender a mathematical experience. Mental operations that support reasoning in mathematics have been posited across time in widely varying theories (Piaget
& Inhelder, 1971; Polya, 1973). Of particular interest to a study of experiences in reasoning within the view tube developments would be forms of reasoning related to measures and measuring specific quantities, using data representations to form conjectures about those quantities, re-presenting those quantities geometrically, analyzing to identify geometric relationships, modeling within a dynamic geometry graphical tool, linking dynamic quantitative measures to graphing representations, and reinterpreting geometrical relationships with algebraic expressions leading to functional interpretations. Each of these would surely constitute a complexity of psychological factors, awaiting careful conceptual analysis.

- Valuing --- One theoretical portrayal of human affect places values at the very core of one’s feelings. As such, these are presumed to be very strongly held beliefs about oneself in the world, probably developed early and derived out of elements of social transmission and enculturation reinforced by one’s lived experiences. As such, both teachers and students exist and function in mathematics classrooms in ways that mirror their idiosyncratically held sense of values. Students today experience school mathematics as an acknowledged, socially important curricular domain, every year. Mathematical experiences, and particularly perceptions of success and achievement, become important aspects of self-concept (and self-esteem or value) within schooling. The quality and nature of student experiences (and particularly success and higher self esteem) can affect the ways “doing mathematics” are, in turn, valued (or de-valued). Thus, one aspect of understanding lived/living mathematical experiences of teachers and students must involve such core feelings of oneself and the subject under study.

- Believing and attitudes --- Other affective elements of lived/living mathematical experience to consider would be the network of beliefs and attitudes held by teacher and students. Beliefs may be seen as anchored in one’s values, but function to influence or explain particular attitudes. One contrast may be to see values and beliefs as more like “traits” of the person, whereas attitudes are “states” and therefore may be more contextualized and variable. Again, in attempting to analyze and interpret the flow of experiences in the view tube scenarios, one should recognize these affective traits and states of participants will likely greatly affect the actual interiorized experiences, but also possible be influenced by the qualities of those lived/living experiences.

At these core aspects of experiential analyses, I’m attempting to portray the closest meanings to a state of “being the stream” (BE). These represent fundamental, idiosyncratic psychological elements of thinking and feeling, and as such would be hypothesized functional mechanisms the mind engages in acts of “knowing” and constructing knowledge. These would be functional, and therefore representational and impactful, in each progressive outward layer of experiential analyses and meaning.

One purpose in presenting my recalled “story” of my classroom view tube episodes is to portray for the reader qualities of potential experiences for both teacher and student in behalf of educational goals that focus on mathematical modeling. From these portrayals and my analyses, I would hope to exemplify possibilities that would stimulate further explorations and conversations toward even deeper understandings of educational possibilities related to this focus on mathematical model building and modeling. Yet, there remain even greater challenges.
Needed Construct Analyses and Research Toward Grounded Theories

My approach as a researcher has been to exhibit my portrayal (as a literary construction) of a composite of my numerous “view tube” teaching episodes, all purposed to engender student experiences related to goals for mathematical models and modeling within a specific context. This construction, and my recalled lived experiences within and across the episodes are my research data. I have used these to develop my analyses and interpretations aimed at revealing what I can see as the multiple layers of meanings that might represent hypothesized “states of being” within the experiential flow. Many questions and issues related to these approaches can be raised (Eisner, 1997; Woo, 2008).

These data would be strengthened, by adding other informed voices to the portrayals, analyses and interpretations. These could include more evidentially based indicators and descriptors of the experiences of students, as they could know those and voice them. Of course, these could be sought within the ongoing experiences as they are occurring, as well as in reflective analysis generated by the students soon after each classroom episode. Additionally, as I noted earlier, the voices of other researcher-participant/observers would add further ideas related to apparent experiences and consequences, albeit as one not fully in the “stream or flow.” Within these observed moments, that researcher-observer could intentionally seek to probe or penetrate more specifically into some of the conjectured layers of meanings explicating above.

A major inherent need within this kind of research involves a lack of clarity and richness related to what can be seen as fundamental psychological and pedagogical constructs that would allow us to work toward more penetrating, grounded theories. It is still the case that most of our educational constructs are derived from long-standing theoretical ideas from various psychologies across time, heavily influenced by epistemologies that presume an objectified, “realist,” structural, “black box,” behavioral and performance outcome conception of mind. By anchoring our research efforts more directly and deeply in studying to understand lived/living classroom experiences of teachers and students, and by adopting more constructively dynamic epistemologies, we may be able to derive new conceptions of those familiar constructs that would allow us to understand phenomenal mathematical experience more deeply.

References


