PARTICIPANT ESSAY


PARTICIPANT RESEARCH ESSAY
FOR QRaMM RESEARCH TEAM

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**Introduction**

In my program of research thus far I have investigated the ways in which reasoning with quantities supports two interrelated components of mathematical activity: creating mathematical generalizations (generalizing) and developing deductive arguments and justifications (proving). Generalizing and proving are considered two of the most critical and fundamental aspects of mathematical activity, and are integrally related to what it means to reason mathematically. In light of the essential role generalizing and proving play in mathematics, it is not surprising that mathematicians educators in the form of recent policy documents have placed an increased emphasis on their role in school mathematics.¹ Broadly construed, in my research I have examined the characterizations of generalizing and proving and the ways that instruction and curriculum can support students’ development of and engagement in these activities. My research agenda has been guided by the following three questions:

¹ A recent RAND study panel of leading mathematicians, mathematics education researchers, and cognitive scientists issued an influential report (*Mathematical Proficiency for All Students: Toward a Strategic Research and Development Program in Mathematics Education*) that identified three target areas for a long-term research and development program. The mathematical practices of generalizing and proving were one of the three target areas. Additionally, the National Council of Teachers of Mathematics (NCTM), the professional organization of both K-12 mathematics teachers and mathematics educators/researchers, in its influential national curriculum standards document (*Principles and Standards for School Mathematics*) elevated proving to one of the primary standards for school mathematics, on the same level as other standards such as algebra.
(1) What are the learning processes that characterize the activities of generalizing and proving? 
(2) In what ways do instructional environments support and constrain students’ abilities to engage in productive generalizing and proving? 
(3) How can mathematics educators more effectively support teachers’ efforts to foster their students’ abilities to generalize and prove?

In the narrative that follows, I will first outline the research I have conducted broadly related to generalization and proof, and then provide more detail about my research on the intersections between these activities and quantitative reasoning.

**Characterizations of Generalizing and Proving**

In my research I investigate the learning processes that support students’ generalizing and proving activities. Much of the work on mathematics students’ abilities to generalize has viewed generalization as the development of a rule (typically formal and algebraic) that serves as a statement about mathematical relationships or properties. Existing frameworks investigating the processes of generalization share a common theme in that generalization is largely viewed as an individual, cognitive construct. More recently, however, views of generalization have expanded to consider the multiple interacting social influences on generalizing. My work is situated within this expanded view of generalization, and attends to how social interactions, tools, personal history, and environments shape people’s generalizing activities. From this perspective I take generalization as a situated practice in which learners engage in at least one of three activities: a) identifying commonality across cases, b) extending one’s reasoning beyond the range in which it originated, or c) deriving broader results from particular cases.

Students’ abilities to create and understand proofs are similarly viewed as a critical component of advanced mathematical activity. Traditionally, the study of proof has been relegated to high school mathematics (particularly the domain of Euclidean geometry) and not addressed again until post-calculus college mathematics. In the last decade, however, this trend began to reverse when the National Council of Teachers of Mathematics (NCTM) released its *Principles and Standards for School Mathematics* which specifically highlighted the importance of deductive reasoning at all levels of mathematics, stating: “Reasoning and proof should be a consistent part of students’ mathematical experience in prekindergarten through grade 12. Reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in many contexts” (NCTM, 2000, p. 56). Mathematics educators have responded to this call by focusing increased attention to research on students’ understanding of proof as well as on teachers’ abilities to support students’ engagement in proving. My research trajectory began at the intersection of these two activities, examining the ways in which engagement in each mutually influences the other.

The framework underlying my research is grounded in the actor-oriented perspective (Lobato, 2003), which occurs at the intersection of radical constructivist and socio-cultural theories. The defining feature of the actor-oriented perspective is its shift from an observer’s (or expert’s) viewpoint to an actor’s (learner’s) viewpoint in order to seek the processes by which students construct meaning in mathematical activities. When examining students’ generalizing activities, the actor-oriented framework highlights the need to understand how learners produce their own relations of similarity between problems. My work therefore investigates what mathematical ideas students view as general, regardless of their appropriateness from the expert’s point of view. Similarly, for examining students’ proofs and justifications, the actor-oriented perspective emphasizes a focus on what students view as convincing, identifying individual student schemes of doubt, truth, and conviction.
I began my research trajectory by examining the relationships between students’ generalizing and justifying activities in algebra. Through a qualitative analysis of middle-school students’ reasoning in two different algebra environments, I developed a taxonomy of students’ generalizations (see Manuscript #4). The taxonomy accounts for multiple levels of generalizing and distinguishes between students’ processes of generalization, or “generalizing activities”, and students’ final statements of generalization, or “reflection generalizations.” The development of this taxonomy provided a way to distinguish between students’ generalizations on multiple dimensions, as well as discern how those types were related to one another. Results indicated that students engaged in powerful cycles of generalizing, developing more sophisticated and productive general statements over time.

Through a continuing analysis of my data corpus, two primary factors have emerged as supports for developing productive generalizations. The first is engaging in acts of justification and proof. I identified links between types of generalizations and associated proof schemes, as well more complex relationships revealing how the two acts mutually influence one another (see Manuscripts #3 and #7). The second factor that emerged is reasoning with quantities and their relationships. In this vein, I identified ways in which students’ abilities to develop emergent quantities influenced their generalizations and justifications when reasoning algebraically (see Manuscripts #1, #5, and #8). In particular, I found that attending to quantities representing constant rates encouraged generalizations about relationships, connections between situations, and dynamic phenomena – this was in contrast to the generalizations about patterns, procedures, and rules that emerged from a number-pattern focus. From these studies I identified the role that emergent-ratio quantities played in supporting the development of powerful generalizations about linear functions.

**Reasoning with Quantities to Support Students’ Abilities to Generalize and Prove**
I have recently begun to focus greater attention on my second and third research questions. In particular, I have broadened my examination of generalizing and proving activities by studying the influence of instructional environments on students’ abilities to productively generalize their learning, as well as construct, understand, and appreciate mathematical proofs. Under the auspices of an NSF RoLE grant I conducted a four-year study to investigate the relationships between the mathematical foci in different teaching environments and students’ abilities to generalize productively about algebraic relationships. Over the course of the project I collected data from four different learning environments in which students engaged in generalizing and proving activities in algebra. The focus of the recent project was twofold, encompassing an examination of the social influence of student learning more generally and a conceptual analysis of student learning of polynomial functions more specifically. To this end I completed an analysis identifying and describing the focusing phenomena in a typical high-school classroom that supported the development of unexpected generalizations about quadratic functions (see Manuscripts #2 and #6). Focusing phenomena are observable features of the classroom environment, such as teachers’ and students’ co-constructed language and gestures, use of artifacts, and interaction with curricula, that regularly direct students’ attention towards certain mathematical properties over others (Lobato, Ellis, & Muñoz, 2003). The construct of focusing phenomena allowed me to account for the ways in which features of the classroom environment influenced how students generalized their learning. The results revealed a focus on counting and calculational techniques that were divorced from any quantitative meaning; these techniques ultimately encouraged a view of the parameter “a” of the function \( y = ax^2 + bx + c \) as the “slope” of the parabola.
The results of that study spurred the development of two teaching experiments, in which I attempted to engineer a more productive mathematical focus on the construction of the relationships between the height and area of growing rectangles in order to develop an understanding of quadratic growth. From the first teaching experiment, I produced a study detailing individual middle-school students’ construction of quantitative relationships to support their generalizations about quadratic functions (see Manuscript #9). From the second teaching experiment, I identified multiple interacting classroom interactions that operated in concert to support students’ generalizing activities (see Manuscript #10). I also recently completed a final analysis of my data corpus from the second teaching experiment to develop a conceptual analysis of students’ learning of quadratic functions (see Manuscript #11). In this analysis I identified ways in which students leveraged their construction of the quantitative relationships between height and area to support three major conceptual advances in their understanding of quadratic growth. This work builds on my earlier work explicating the role that emergent quantities played in supporting students’ productive generalizations about linear functions.

The identification of the ways in which reasoning with quantities has been shown to foster students’ generalizing activities led to the development of my most recent NSF grant (CAREER), in which I detailed the potential ways in which reasoning with quantities can support the development of students’ proof practices. As I continue to move forward in investigating the processes of proving with this newest grant, I will conduct a five-year study beginning in September 2010 to explore how constructing quantities and reasoning with quantitative relationships supports middle-school students’ abilities to engage in deductive argument and proof. Middle school is a particularly critical time for students to develop their proof practices because it marks a shift from the concrete, arithmetic reasoning of elementary school mathematics to a focus on the increasingly complex, abstract reasoning required for high school mathematics. For the first two years of the grant, I will conduct a series of teaching experiments with middle-school students in the domains of linear, quadratic, and exponential functions, exploring how a focus on quantitative relationships supports students’ proving activities. During the latter half of the study, I will then shift my attention to my third research question to investigate ways to support teachers’ abilities to foster their own students’ proof practices through the use of quantitative reasoning. I will implement professional development opportunities for practicing teachers in order to communicate the project’s findings on the use of quantities to support students’ proof practices. After the professional development intervention, I will collect data from participating teachers’ classrooms in order to study the teachers’ effectiveness in using what they have learned to foster their students’ proof skills. This final phase represents a scaling up of my research agenda, shifting from previously conducted smaller-scale studies to implementing and studying the effects of whole-class interventions.

References


Relevant Manuscripts Mentioned Above