THE C*-ALGEBRA OF A MINIMAL HOMEOMORPHISM OF ZERO MEAN DIMENSION

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Abstract. Let $X$ be an infinite compact metrizable space, and let $\sigma : X \to X$ be a minimal homeomorphism. Suppose that $(X, \sigma)$ has zero mean topological dimension. The associated C*-algebra $A = C(X) \rtimes_{\sigma} \mathbb{Z}$ is shown to absorb the Jiang-Su algebra $\mathcal{Z}$ tensorially, i.e., $A \cong A \otimes \mathcal{Z}$. This implies that $A$ is classifiable when $(X, \sigma)$ is uniquely ergodic.

Moreover, without any assumption on the mean dimension, it is shown that $A \otimes A$ always absorbs the Jiang-Su algebra.

1. Introduction

Recently, Toms and Winter proved that a simple C*-algebra arising from a $\mathbb{Z}$-action on a compact metrizable space of finite dimension absorbs the Jiang-Su C*-algebra $\mathcal{Z}$ ([16], [17]). (This definitive result followed much earlier work, e.g., [7].) As shown in [3], some condition is necessary. (Presumably, mean dimension zero!)

In the present note we show that the condition of finite dimension can be replaced by the weaker condition that the dynamical system have mean dimension zero, as defined in [10] (Definition 2.1 below): More precisely,

Theorem. Let $X$ be an infinite compact metrizable space, and let $\sigma : X \to X$ be a minimal homeomorphism. If $(X, \sigma)$ has mean dimension zero, then the C*-algebra $A = C(X) \rtimes_{\sigma} \mathbb{Z}$ absorbs the Jiang-Su algebra $\mathcal{Z}$ tensorially.

The same classification consequences as shown in [16] and [17] in the case that $K_0$ separates traces hold also in the present setting. See Corollary 4.7. In particular, it follows that the C*-algebra of any uniquely ergodic dynamical system is classifiable (since in this case the mean dimension is automatically zero).

Moreover, the tensor product of the C*-algebras of two arbitrary minimal homeomorphisms (without any assumption on the mean dimension) is Jiang-Su stable:

Theorem. Let $(X_1, \sigma_1)$ and $(X_2, \sigma_2)$ be minimal dynamical systems, where $X_1$ and $X_2$ are infinite compact metrizable spaces. Consider the C*-algebras

$$A_1 = C(X_1) \rtimes_{\sigma_1} \mathbb{Z} \quad \text{and} \quad A_2 = C(X_2) \rtimes_{\sigma_2} \mathbb{Z}.$$  

Then

$$A_1 \otimes A_2 \cong (A_1 \otimes A_2) \otimes \mathcal{Z}.$$  

2. Mean topological dimension and the small boundary property

Let $X$ be a compact metrizable space, and let $\sigma : X \to X$ be a homeomorphism. (These objects will be fixed throughout the paper.)
Definition 2.1 ([10]). The mean topological dimension of \((X, \sigma)\), denoted by \(\text{mdim}(X, \sigma)\), is defined by

\[
\text{mdim}(X, \sigma) = \sup_{\alpha} \lim_{N \to \infty} \frac{1}{N} D(\alpha \lor \sigma(\alpha) \lor \cdots \lor \sigma^{N-1}(\alpha)),
\]

where the dimension of the finite open cover \(\beta, D(\beta)\), is the number \(\min\{\text{ord}(\beta'); \beta' \preceq \beta\}\). (By the order of a cover \(\beta\) is meant the number \(\text{ord}(\beta) = -1 + \sup_x \sum_{U \in \beta} \chi_U(x)\).)

Definition 2.2 ([10]). For each set \(E \subseteq X\), the orbit capacity of \(E\), denoted by \(\text{ocap}(E)\), is defined to be

\[
\text{ocap}(E) = \lim_{N \to \infty} \frac{1}{N} \sup \{\chi_E(x) + \cdots + \chi_E(\sigma^{N-1}(x)); x \in X\}.
\]

The system \((X, \sigma)\) is said to have the small boundary property (SBP) if for any \(x \in X\) and any open neighborhood \(U\) of \(x\), there is a neighborhood \(V\) in \(U\) such that \(\text{ocap}(\partial V) = 0\).

Theorem 2.3 ([10], [9]). If \(\sigma\) is minimal, then \((X, \sigma)\) has zero mean topological dimension if and only if it has the small boundary property.

Proposition 2.4 (Proposition 5.3 of [10]). If \((X, T)\) has the SBP, then for every open cover \(\alpha\) of \(X\) and every \(\varepsilon > 0\), there is a partition of unity \(\phi_j : X \to [0, 1]\) \((j = 1, \ldots, |\alpha|)\) subordinate to \(\alpha\) such that

\[
\text{ocap}\left(\bigcup_{j=1}^{|\alpha|} \phi_j^{-1}((0,1))\right) < \varepsilon.
\]

3. The C*-algebra of a homeomorphism and its large subalgebras

Suppose that \(X\) as above is an infinite set and \(\sigma\) as above is minimal. Let us denote by \(\sigma\) also the automorphism of \(C(X)\) defined by

\[
\sigma(f) = f \circ \sigma^{-1}, \quad \forall f \in C(X).
\]

Consider the crossed product C*-algebra

\[
A = C(X) \rtimes_\sigma \mathbb{Z} = C^*(f, u; \ ufu^* = \sigma(f), \ f \in C(X)).
\]

Fix \(y \in X\), and then consider the sub-C*-algebra

\[
A_y = C^*(f, ug; \ f, g \in C(X), \ g(y) = 0) \subseteq A.
\]

Let \(Y\) be a closed neighborhood of \(y\) in \(X\). Consider the sub-C*-algebra

\[
A_Y = C^*(f, ug; \ f, g \in C(X), \ g|_Y = 0) \subseteq A_y.
\]

It is clear that \(A_{Y_1} \subseteq A_{Y_2}\) if \(Y_1 \supseteq Y_2\), and \(A_y\) is the inductive limit of \(A_{Y_i}\) if \(\bigcap Y_i = \{y\}\).

Consider the first return times

\[
\{j \in \mathbb{N} \cup \{0\}; \ \sigma^j(x) \in Y \text{ but } \sigma^i(x) \notin Y, \ 1 \leq i \leq j - 1 \text{ for some } x \in Y\}.
\]

Since \(\sigma\) is minimal and \(X\) is compact, this set of numbers is finite; let us write it as

\[
J_1 < J_2 < \cdots < J_K
\]

for some \(K \in \mathbb{N}\). Note that since \(X\) is an infinite set and \(\sigma\) is minimal, the first return time \(J_1\) can be arbitrarily large if \(Y\) is sufficiently small.
For each $1 \leq k \leq K$, consider the (locally compact—see below) subset of $X$

$$Z_k = \{ x \in Y; \sigma^i(x) \in Y \text{ but } \sigma^i(x) \notin Y \text{ for any } 1 \leq i \leq J_k - 1 \}.$$

Then the sets

$$\{\{Z_1, \sigma(Z_1), \ldots, \sigma^{J_1-1}(Z_1)\}, \ldots, \{Z_k, \sigma(Z_k), \ldots, \sigma^{J_k-1}(Z_k)\}\}$$

(which are naturally grouped as shown) form a partition of $X$. This is often called a Rokhlin partition.

**Lemma 3.1** ([8]). With notation as above, one has that, for each $1 \leq k \leq K$,

1. the set $Z_1 \cup \cdots \cup Z_k$ is closed (and so $Z_k$ is locally compact),
2. the set $\overline{Z_k} \cap (Z_1 \cup \cdots \cup Z_{k-1})$ is the disjoint union of the subsets

$$W = \partial Z_k \cap Z_{t_1} \cap \sigma^{-J_{t_1}}(Z_{t_2}) \cap \cdots \cap \sigma^{-(J_{t_1} + \cdots + J_{t_s} - 1)}(Z_{t_s}),$$

where $J_{t_1} + \cdots + J_{t_{s-1}} + J_{t_s} = J_k$.

A fairly explicit description of the subalgebra $A_{Y}$ of the crossed product, which in fact is a $C^*$-algebra of type I, was obtained by Q. Lin ([8]). It is a subhomogeneous algebra, of order at most $J_k$.

**Theorem 3.2** ([8]). With notation as above, one has that the $C^*$-algebra $A_{Y}$ is isomorphic to the sub-$C^*$-algebra of $\bigoplus_{k=1}^{K} M_{J_k}(C(\overline{Z_k}))$ consisting of the elements $(F_1, \ldots, F_k)$ with

$$F_k|_W = \begin{pmatrix} F_{t_1}|_W & F_{t_2} \circ \sigma^{J_{t_1}}|_W & \cdots & F_{t_s} \circ \sigma^{J_{t_s} - 1}|_W \end{pmatrix}$$

whenever

$$W = \partial Z_k \cap Z_{t_1} \cap \sigma^{-J_{t_1}}(Z_{t_2}) \cap \cdots \cap \sigma^{-(J_{t_1} + \cdots + J_{t_s} - 1)}(Z_{t_s}) \neq \emptyset,$$

where $J_{t_1} + \cdots + J_{t_{s-1}} + J_{t_s} = J_k$.

Moreover, for any $f, g \in C(X)$ with $g|_Y = 0$, the images of $f, ug \in A_{Y}$ in this identification are

$$f = \bigoplus_{k=1}^{K} \begin{pmatrix} f \circ \sigma|_{\overline{Z_k}} & f \circ \sigma^2|_{\overline{Z_k}} & \cdots & f \circ \sigma^{J_k}|_{\overline{Z_k}} \end{pmatrix} \in \bigoplus_{k=1}^{K} M_{J_k}(C(\overline{Z_k}))$$

and

$$ug = \bigoplus_{k=1}^{K} \begin{pmatrix} 0 & g \circ \sigma|_{\overline{Z_k}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & g \circ \sigma^{J_k-1}|_{\overline{Z_k}} \end{pmatrix} \in \bigoplus_{k=1}^{K} M_{J_k}(C(\overline{Z_k})),$$

respectively.

The sub-$C^*$-algebra $A_y$ in $A$ is a typical example of a large sub-$C^*$-algebra.
Definition 3.3 ([13], [1]). Let $A$ be an infinite dimensional simple separable unital C*-algebra. A unital sub-C*-subalgebra $B \subseteq A$ is said to be large in $A$ if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, ..., a_m \in A$, $\varepsilon > 0$, $x \in A^+$ with $\|x\| = 1$, and $y \in B^+ \setminus \{0\}$, there are $c_1, c_2, ..., c_m \in A$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, ..., m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, ..., m$ we have $(1 - g)c_j, c_j(1 - g) \in B$.
4. $g \preceq B y$.
5. $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

Moreover, if

6. for $j = 1, 2, ..., m$ we have $\|ga_j - a_jg\| < \varepsilon$,

then the sub-C*-algebra $B$ is said to be centrally large in $A$.

Theorem 3.4 (Archey-Phillips [1]). The C*-algebra $A_y$ is centrally large in $A$.

Theorem 3.5 (Archey-Phillips [1]). If $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$ are centrally large sub-C*-algebras, then the tensor product sub-C*-algebra

$$B_1 \otimes_{\min} B_2 \subseteq A_1 \otimes_{\min} A_2$$

is centrally large in the tensor product.

We will use the following property of centrally large sub-C*-algebras.

Theorem 3.6 (Archey-Phillips [1]). Let $B \subseteq A$ be a nuclear centrally large sub-C*-algebra of $A$. If $B \cong B \otimes \mathbb{Z}$, then $A \cong A \otimes \mathbb{Z}$.

4. The C*-Algebra of a Minimal Homeomorphism of Mean Dimension Zero

Let $S$ be a subhomogeneous C*-algebra, with dimensions of irreducible representations $d_1 < d_2 < \cdots < d_n$. The dimension ratio of $S$ is defined as

$$\dimRatio(S) = \max\left\{\frac{\dim(\text{Prim}_{d_1}(S))}{d_1}, \frac{\dim(\text{Prim}_{d_2}(S))}{d_2}, ..., \frac{\dim(\text{Prim}_{d_n}(S))}{d_n}\right\},$$

where $\dim(\cdot)$ denotes the topological covering dimension.

By Proposition 2.13 (together with 2.5 and 2.9) of [12], if the primitive ideal spaces of $S$ have finite dimension, then the C*-algebra $S$ has a recursive subhomogeneous decomposition,

$$S \cong \left[\cdots \left[C_0 \oplus_{C_i^{(0)}} C_1 \oplus_{C_i^{(0)}} C_2 \cdots \right] \oplus_{C_i^{(0)}} C_{l}\right],$$

with $C_k = C(X_k, M_{n(k)})$ for compact Hausdorff spaces $X_k$ and positive integers $n(k)$, and with $C_k^{(0)} = C(X_k^{(0)}, M_{n(k)})$ for compact subsets $X_k^{(0)} \subseteq X_k$ (possibly empty) such that

$$\frac{\dim(X_k)}{n(k)} \leq \dimRatio(S), \quad 0 \leq k \leq l.$$

(See [12] for more details on recursive subhomogeneous C*-algebras.)

In this section, it will be shown (using Theorem 3.2 indirectly) that if $(X, \sigma)$ has zero mean dimension (and, as understood, $\sigma$ is minimal), then the large subalgebra $A_y$ can be locally approximated by subhomogeneous C*-algebras with arbitrarily small dimension ratio (see Theorem 4.4).
As a consequence of this, it follows (on applying the large subalgebra technique—see [13]) that the crossed product C*-algebra $C(X) \rtimes_\sigma \mathbb{Z}$ absorbs the Jiang-Su algebra $\mathcal{Z}$, the main result of this paper.

Of the following three lemmas (Lemmas 4.1, 4.2, and 4.3), only the first concerns dynamical systems. The other two are elementary C*-algebra results, at least the second of which, a case of the Stone-Weierstrass Theorem, is known.

**Lemma 4.1.** Let $Y \subseteq X$ be a closed subset with nonempty interior. Denote by $Z_1, ..., Z_K$ the bases of the Rokhlin towers generated by $Y$, and by $J_1 < J_2 < \cdots < J_K$ the first return times of $Z_1, Z_2, ..., Z_K$, respectively. There is an open set $U \supseteq Y$ such that for each $1 \leq k \leq K$, one has

$$\frac{1}{J_k}(\chi_U(x) + \chi_U(\sigma(x)) + \cdots + \chi_U(\sigma^{J_k-1}(x))) \leq \frac{1}{J_1}, \quad x \in Z_k.$$

**Proof.** Note that by definition the inequality holds with $Y$ in place of $U$. (So, the question is to extend this in some sense by continuity to a neighborhood—we propose to do this by induction on $k$.)

Since $Y$ is closed, and the sets

$$Y, \sigma(Y), ..., \sigma^{J_1-1}(Y)$$

are pairwise disjoint, there is an open set $U \supseteq Y$ such that

$$U, \sigma(U), ..., \sigma^{J_1-1}(U)$$

are pairwise disjoint. In particular,

$$\frac{1}{J_1}(\chi_U(x) + \chi_U(\sigma(x)) + \cdots + \chi_U(\sigma^{J_1-1}(x))) \leq \frac{1}{J_1}, \quad x \in Z_1.$$

Let $2 \leq k \leq K$, and assume that we have constructed an open set $U \supseteq Y$ such that for any $1 \leq i \leq k-1$,

$$\frac{1}{J_i}(\chi_U(x) + \chi_U(\sigma(x)) + \cdots + \chi_U(\sigma^{J_i-1}(x))) \leq \frac{1}{J_1}, \quad x \in Z_i. \quad (4.1)$$

Let us construct another open neighborhood of $Y$, still to be denoted by $U$ (just shrink!), such that $(4.1)$ holds for $i = k$.

First, pick an open neighborhood $U'$ of $Y$ such that $U' \subseteq U$. Let $x \in \overline{Z_k} \cap (Z_1 \cup \cdots \cup Z_{k-1})$. If $x \in W = \overline{Z_k} \cap Z_{t_1} \cap \sigma^{-J_{t_1}}(Z_{t_2}) \cap \cdots \cap \sigma^{-(J_{t_1} + \cdots + J_{t_s}-1)}(Z_{t_s})$, where $J_{t_1} + \cdots + J_{t_{s-1}} + J_{t_s} = J_k$, then the orbit of $x$ is

$$x, \sigma(x), ..., \sigma^{J_{t_1}-1}(x), \sigma^{J_{t_1}}(x), ..., \sigma^{J_{t_2}}(\sigma^{J_{t_1}}(x)), ..., \sigma^{J_1+\cdots+J_{t_s-1}}(x), ..., \sigma^{J_s}(\sigma^{J_{t_1}+\cdots+J_{t_s-1}}(x)).$$

By the induction hypothesis (4.1), one has

$$\frac{1}{J_k}(\chi_U(x) + \chi_U(\sigma(x)) + \cdots + \chi_U(\sigma^{J_k-1}(x))) \leq \frac{1}{J_1},$$

and therefore, there is a neighborhood $V_x$ of $x$ such that

$$\frac{1}{J_k}(\chi_U(z) + \chi_U(\sigma(z)) + \cdots + \chi_U(\sigma^{J_k-1}(z))) \leq \frac{1}{J_1}, \quad z \in V_x. \quad (4.2)$$
Hence, there is an open set $E$ such that

$$\overline{Z_k} \cap (Z_1 \cup \cdots \cup Z_{k-1}) \subseteq E$$

and

$$\frac{1}{J_k} (\chi_{U'}(z) + \chi_{U'}(\sigma(z)) + \cdots + \chi_{U'}(\sigma^{k-1}(z))) \leq \frac{1}{J_1}, \quad z \in E. \quad (4.3)$$

Replace $U$ by $U'$ and still denote it by $U$. Since $Z_1 \cup \cdots \cup Z_{k-1} \cup Z_k$ is a closed set, one has that

$$\overline{Z_k} \setminus Z_k \subseteq \overline{Z_k} \cap (Z_1 \cup \cdots \cup Z_{k-1}),$$

and hence $\overline{Z_k} \setminus Z_k \subseteq E$. In particular,

$$\overline{Z_k} \setminus E = Z_k \setminus E,$$

and $Z_k \setminus E$ is a compact set.

For any point $x$ in $Z_k \setminus E$, one can shrink $U$ further so that

$$\frac{1}{J_k} (\chi_U(x) + \chi_U(\sigma(x)) + \cdots + \chi_U(\sigma^{k-1}(x))) \leq \frac{1}{J_1}. \quad (4.4)$$

Note that (4.4) holds for a neighborhood of $x$. Since $Z_k \setminus E$ is compact, there is an open neighborhood $U$ of $Y$ such that

$$\frac{1}{J_k} (\chi_U(x) + \chi_U(\sigma(x)) + \cdots + \chi_U(\sigma^{k-1}(x))) \leq \frac{1}{J_1}, \quad x \in Z_k \setminus E. \quad (4.5)$$

Together with (4.3), one has

$$\frac{1}{J_k} (\chi_U(x) + \chi_U(\sigma(x)) + \cdots + \chi_U(\sigma^{k-1}(x))) \leq \frac{1}{J_1}, \quad x \in Z_k, \quad (4.6)$$

as desired. \hfill \Box

**Lemma 4.2.** Consider $n \times n$ matrices

$$A := \text{diag}\{a_1, \ldots, a_n\}, \quad B := \text{diag}\{b_1, \ldots, b_n\}$$

$$C := \begin{pmatrix} 0 & c_1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{n-1} \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \quad \text{and} \quad D := \begin{pmatrix} 0 & 0 & \cdots & 0 \\ d_1 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & d_{n-1} \end{pmatrix},$$

where $0 < c_i, d_i \leq 1$. If the pair $(A, C)$ is unitarily equivalent to pair $(B, D)$, then

$$a_i = b_i, \quad c_j = d_j, \quad 1 \leq i \leq n, 1 \leq j \leq n - 1.$$

**Proof.** Let $W \in M_n(\mathbb{C})$ be a unitary such that

$$W^*AW = B \quad \text{and} \quad W^*CW = D.$$ 

For each $1 \leq k \leq n$, one has $W^*((C^*)^k C^k)W = (D^*)^k D^k$, and a functional calculus argument shows that

$$W^*(e_1 + \cdots + e_k)W = e_1 + \cdots + e_k, \quad 1 \leq k \leq n,$$

where $e_i$ is the $i$th standard rank-one projection. This implies that

$$W^*e_iW = e_i, \quad 1 \leq i \leq n.$$
Since $W^*AW = B$, it follows that
$$W^*e_iAe_iW = e_iBe_i, \quad 1 \leq i \leq n,$$
and hence
$$a_i = b_i, \quad 1 \leq i \leq n.$$
A similar argument shows that $c_i = d_i, \quad 1 \leq i \leq n$. □

**Lemma 4.3.** Let $Z$ be a second countable locally compact Hausdorff space, and let $S$ be a sub-C*-algebra of $M_n(C_0(Z))$. Suppose that there is a surjective continuous map $\xi : Z \to \Delta$ such that

1. $\xi(x_1) = \xi(x_2)$ if and only if $\pi_{x_1}|S$ is unitarily equivalent to $\pi_{x_2}|S$,
2. for any $g \in S$, if $\xi(x_n) \to \xi(x)$, then $g(x_n) \to g(x)$, and
3. $\pi_x(S) = M_n(\mathbb{C})$, for any $x \in Z$.

Then $S \cong M_n(C_0(\Delta))$.

**Proof.** For each $f \in S$, define a function $\tilde{f} : \Delta \to M_n(\mathbb{C})$ by
$$\tilde{f}(z) = f(x), \quad \text{if } \xi(x) = z.$$ 

By Condition (2), $\tilde{f}$ is well defined, and $\tilde{f}$ is continuous. Moreover, $\tilde{f}$ vanishes at infinity. As if $z_n \in \Delta$ with $z_n \to \infty$, since $\xi$ is surjective, there are $x_n \in Y$ with $\xi(x_n) = z_n$. Then $x_n \to \infty$. Otherwise, there is a subsequence, say $(x_{n_j})$, converging to a point $x \in Z$. Since $\xi$ is continuous, one has that $z_{n_j} = \xi(x_{n_j}) \to \xi(x)$, which contradicts the assumption $z_n \to \infty$. Hence $\tilde{f}(z_n) = f(x_n) \to 0$, and $\tilde{f} \in M_n(C_0(\Delta))$.

Moreover, it is clear that the map $f \to \tilde{f}$ is an injective homomorphism, and thus one can regard $S$ as a sub-C*-algebra of $M_n(C_0(\Delta))$. It follows from Conditions (1) and (3) that $S$ is a rich sub-C*-algebra of $M_n(C_0(\Delta))$ in the sense of Dixmier (11.1.1 of [2]), and therefore $S = M_n(C_0(\Delta))$ by Proposition 11.1.6 of [2] (or, it follows from Theorem 7.2 of [4]). □

**Theorem 4.4.** Let $X$ be an infinite compact metrizable space, and let $\sigma$ be a minimal homeomorphism. Suppose that $(X, \sigma)$ has topological mean dimension zero. Let
$$\{f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_m\} \subseteq C(X)$$
with $g_i(W) = \{0\}, \ i = 1, \ldots, m$, for some open set $W$ containing $y$. Then, for any $\varepsilon > 0$, there is a closed neighborhood $Y$ of $y$ contained in $W$ such that the finite subset
$$\{f_1, f_2, \ldots, f_n, ug_1, ug_2, \ldots, ug_m\}$$
of $A_Y$, where $u$ is the canonical unitary of the crossed product, is approximated to within $\varepsilon$ by a subhomogeneous C*-algebra $S$ in $A_Y$ with dimension ratio at most $\varepsilon$.

**Proof.** Let $\varepsilon > 0$ be arbitrary. Choose a finite open cover
$$\alpha = \{U_1, U_2, \ldots, U_{|\alpha|}\}$$
of $X$ such that
$$|f_i(x) - f_i(y)| < \varepsilon \quad \text{and} \quad |g_j(x) - g_j(y)| < \varepsilon, \quad x, y \in U_i, \ 1 \leq i \leq |\alpha|.$$  

(4.7)
Since \((X, \alpha)\) is minimal and has mean dimension zero, it has SBP, and therefore by Proposition 2.4, there is a partition of unity \(\{\phi_U; U \in \alpha\}\) subordinate to \(\alpha\) and \(T \in \mathbb{N}\) such that
\[
\frac{1}{N}(\chi_E(x) + \chi_E(\sigma(x)) + \cdots + \chi_E(\sigma^{N-1}(x))) < \frac{\varepsilon}{|\alpha| + 1}, \quad x \in X, \ N \geq T,
\]
where \(E = \bigcup_{U \in \alpha} \phi_U^{-1}((0, 1))\).

Choose the closed neighborhood \(Y\) of \(y\) in \(W\) as follows: the Rokhlin partition
\[
\{\{Z_1, \sigma(Z_1), \ldots, \sigma^{J_1-1}(Z_1)\}, \ldots, \{Z_K, \sigma(Z_k), \ldots, \sigma^{J_K-1}(Z_k)\}\}
\]
corresponding as in Section 3 to \(Y\) should satisfy
\[
J_1 \geq \max\{\frac{|\alpha| + 1}{\varepsilon}, T\}.
\]

By Lemma 4.1, there is an open set \(V\) such that \(Y \subseteq V\), and for any \(1 \leq k \leq K\),
\[
\frac{1}{J_k}(\chi_V(x) + \chi_V(\sigma(x)) + \cdots + \chi_V(\sigma^{J_k-1}(x))) \leq \frac{1}{J_1} < \frac{\varepsilon}{|\alpha| + 1}, \quad x \in Z_k.
\]

Choose a continuous function \(H : X \to [0, 1]\) such that
\[
H^{-1}(0) = Y \quad \text{and} \quad H^{-1}(1) \supseteq (X \setminus V).
\]

Since \(Y \subseteq W\), without loss of generality, we may assume that \(V \subseteq W\), and then
\[Hg_j = g_j, \quad 1 \leq j \leq m.\]

Let us show that the sub-C*-algebra
\[S := C^*\{\phi_U, uH; U \in \alpha\} \subseteq A_Y,
\]
together with the closed set \(Y\), satisfies the conditions of the theorem.

For each \(U \in \alpha\), pick a point \(x_U \in U\). Then, by (3.1), for each \(f_i, 1 \leq i \leq n\), one has
\[
\left\|f_i - \sum_{U \in \alpha} f_i(x_U)\phi_U\right\| \leq \sup_{x \in X} \sum_{U \in \alpha} |f(x) - f_i(x_U)| \phi_U(x) < \varepsilon;
\]
and for each \(g_j, 1 \leq j \leq m\), one has
\[
\left\|u g_j - uH \sum_{U \in \alpha} g_j(x_U)\phi_U\right\| = \left\|uHg_j - uH \sum_{U \in \alpha} g_j(x_U)\phi_U\right\|
\leq \left\|g_j - \sum_{U \in \alpha} g_j(x_U)\phi_U\right\| < \varepsilon.
\]

This shows the approximate inclusion
\[\{f_1, f_2, \ldots, f_n, u g_1, u g_2, \ldots, u g_m\} \subseteq \varepsilon S.
\]

Finally, let us show that \(\dimRatio(S) < \varepsilon\). For each \(1 \leq k \leq K\), consider the algebra
\[M_{J_k}(C(Z_k))\]
of Theorem 3.2 and consider the map
\[\xi_k : Z_k \to \mathbb{R}^{(|\alpha| + 1)J_k - 1}\]
defined by
\begin{equation}
(4.10) \quad \xi_k(x) \mapsto ((\Phi \circ \sigma(x), \Phi \circ \sigma^2(x), \ldots, \Phi \circ \sigma^{J_k}(x)), (H \circ \sigma(x), \ldots, H \circ \sigma^{J_k-1}(x))),
\end{equation}
where the map $\Phi : \mathcal{Z}_k \to \mathbb{R}^{[\alpha]}$ is defined by
\[ \Phi = \bigoplus_{U \in \alpha} \phi_U. \]

By (4.8) and (4.9), the image of $Z_k$ under $\xi_k$ is contained in the set
\[ \{(t_1, t_2, \ldots, t_{[\alpha]+1}, J_k-1) \in [0, 1]^{[\alpha]+1} J_k; \text{ at most } \varepsilon J_k \text{ of the } t_i \text{ are not } 0 \text{ or } 1\}, \]
which has dimension at most $\varepsilon J_k - 1$ (as it is a union of simplices with at most $\varepsilon J_k$ vertices). Therefore, $\xi_k(Z_k)$ has dimension at most $\varepsilon J_k$. For convenience, write $\xi_k(Z_k) = \Delta_k$. We have
\[ \dim(\Delta_k) < \varepsilon J_k. \]

For each $x \in Z_k$, the evaluation map $\pi_x$ on $A_Y$ is an irreducible representation of $A_Y$ with dimension $J_k$. Consider the restriction of $\pi_x$ to $S$. Note that for any $x_1, x_2 \in Z_k$, if
\[ \xi_k(x_1) = \xi_k(x_2), \]
then, by the definition of $\xi_k$, one has
\[ \phi_U \circ \sigma^i(x_1) = \phi_U \circ \sigma^i(x_2) \quad \text{and} \quad H \circ \sigma^j(x_1) = H \circ \sigma^j(x_2), \]
where $U \in \alpha, 1 \leq i \leq J_k, 1 \leq j \leq J_k - 1$. By (3.1) and (3.2) of Theorem 3.2 one has that
\[ \pi_{x_1}(\phi_U) = \pi_{x_2}(\phi_U) \quad \text{and} \quad \pi_{x_1}(uH) = \pi_{x_2}(uH), \quad U \in \alpha. \]

Since $S$ is the sub-C*-algebra generated by $\phi_U, U \in \alpha$, and $uH$, one has
\begin{equation}
(4.11) \quad \pi_{x_1}|S = \pi_{x_2}|S.
\end{equation}

Moreover, for any $g \in S, x \in Z_k$, and any sequence $(x_n)$ in $Z_k$, if $\xi_k(x_n) \to \xi_k(x)$, then
\begin{equation}
(4.12) \quad \pi_{x_n}(g) \to \pi_x(g).
\end{equation}

For any $x \in Z_k$, the representation $\pi_x|S$ is irreducible on $S$ (hence has dimension $J_k$). In fact, let us consider the image of $uH$ under $\pi_x$, which is
\[ w := \begin{pmatrix}
0 \\
H(\sigma(x)) & 0 \\
\vdots & \ddots \\
& \ddots & H(\sigma^{J_k-1}(x)) & 0
\end{pmatrix} \in \pi_x(S). \]

Noting that $H^{-1}(0) = Y$ and $x \in Z_k$, one has
\begin{equation}
(4.13) \quad H(\sigma^i(x)) \neq 0, \quad 1 \leq i \leq J_k - 1.
\end{equation}

Then the C*-algebra generated by $w$ is the full matrix algebra $M_{J_k}(\mathbb{C})$, and the restriction of $\pi_x$ to $S$ must be irreducible. In particular,
\begin{equation}
(4.14) \quad \pi_x(S) = M_{J_k}(\mathbb{C}).
\end{equation}

Therefore, one has that the dimension of an irreducible representation of $S$ must be $J_k$ for some $k$, and each irreducible representation of $S$ with dimension $J_k$ is the restriction of $\pi_x$ for some $x \in Z_k$. 
Let \( x_1, x_2 \in Z_k \). One asserts that
\[
(4.15) \quad \pi_{x_1}|_S \text{ and } \pi_{x_2}|_S \text{ are unitarily equivalent if and only if } \xi_k(x_1) = \xi_k(x_2).
\]
If \( \xi_k(x_1) = \xi_k(x_2) \), then, as shown above, one has
\[
\pi_{x_1}|_S = \pi_{x_2}|_S.
\]
In particular, \( \pi_{x_1}|_S \) and \( \pi_{x_2}|_S \) are unitarily equivalent.

Now, assume that \( \pi_{x_1}|_S \) and \( \pi_{x_2}|_S \) are unitarily equivalent. Pick \( \phi_U \), and consider the pair \( (\phi_U, uH) \). Again, by (3.1) and (3.2) of Theorem 3.2 one has that
\[
\pi_{x_1}(\phi_U) = \begin{pmatrix} \phi_U(\sigma(x_1)) \\ & \ddots \\ & & \phi_U(\sigma^{J_k-1}(x_1)) \end{pmatrix}, \quad \pi_{x_1}(uH) = \begin{pmatrix} 0 \\ H(\sigma(x_1)) \\ & \ddots \\ & & H(\sigma^{J_k-1}(x_1)) \end{pmatrix},
\]
and
\[
\pi_{x_2}(\phi_U) = \begin{pmatrix} \phi_U(\sigma(x_2)) \\ & \ddots \\ & & \phi_U(\sigma^{J_k-1}(x_2)) \end{pmatrix}, \quad \pi_{x_2}(uH) = \begin{pmatrix} 0 \\ H(\sigma(x_2)) \\ & \ddots \\ & & H(\sigma^{J_k-1}(x_2)) \end{pmatrix}.
\]
Since \( \pi_{x_1} \) and \( \pi_{x_2} \) are assumed to be unitarily equivalent, the pair of matrices \( (\pi_{x_1}(\phi_U), \pi_{x_1}(uH)) \) is unitarily equivalent to the pair of matrices \( (\pi_{x_2}(\phi_U), \pi_{x_2}(uH)) \). By (4.13), one may apply Lemma 4.2 to obtain
\[
\phi_U(\sigma^i(x_1)) = \phi_U(\sigma^i(x_2)) \quad \text{and} \quad H(\sigma^i(x_1)) = H(\sigma^i(x_2)), \quad 1 \leq i \leq J_k, 1 \leq j \leq J_k - 1.
\]
Applying this argument for all \( U \in \alpha \), one has that
\[
\phi_U(\sigma^i(x_1)) = \phi_U(\sigma^i(x_2)) \quad \text{and} \quad H(\sigma^j(x_1)) = H(\sigma^j(x_2)), \quad U \in \alpha, 1 \leq i \leq J_k, 1 \leq j \leq J_k - 1,
\]
and this implies (by the construction of the map \( \xi_k \); see (4.10))
\[
\xi_k(x_1) = \xi_k(x_2).
\]
This proves the assertion.

Since any irreducible representation of \( S \) is contained in a irreducible representation of \( A_Y \), and \( \{\pi_x; x \in Y\} \) are all of the irreducible representations of \( A_Y \), one has that the dimensions of the irreducible representations of \( S \) have to be \( J_1, J_2, \ldots, J_K \), and the map \( \xi \) induces a bijection between \( \text{Prim}_{k}(S) \) and \( \Delta_k \) for each \( 1 \leq k \leq K \). Then the subquotient with \( J_k \)-dimensional representations of \( S \), denoted by \( S_k \), is a sub-C*-algebra of the subquotient with \( J_k \)-dimensional representations of \( A_Y \), which is canonically isomorphic to \( M_{J_k}(C_0(Z_k)) \). By (4.15), one has that for any \( x_1, x_2 \in Z_k \),
\[
(4.16) \quad \pi_{x_1}|_S \text{ is unitarily equivalent to } \pi_{x_2}|_S \text{ if and only if } \xi_k(x_1) = \xi_k(x_2).
\]
By (4.12), one has that for any \( g \in S_k \), any \( x \in Z_k \), and any sequence \( (x_n) \) in \( Z_k \), if \( \xi_k(x_n) \to \xi_k(x) \), then
\[
(4.17) \quad \pi_{x_n}(g) \to \pi_x(g).
\]
Therefore, the conditions of Lemma 4.3 are satisfied for the sub-C*-algebra $S_k$ of $M_{J_k}(C_0(Z_k))$, and it follows that

$$S_k \cong M_{J_k}(C_0(\Delta_k)).$$

This implies that

$$\text{Prim}_{J_k}(S) = \text{Prim}(S_k) = \Delta_k,$$

and hence

$$\dim(\text{Prim}_{J_k}(S)) = \dim(\Delta_k) < \varepsilon J_k.$$ 

In particular, one has

$$\dim\text{Ratio}(S) < \varepsilon,$$

as desired. \hfill \Box

**Theorem 4.5.** Let $X$ be an infinite compact metrizable space, and let $\sigma : X \to X$ be a minimal homeomorphism. If $(X,\sigma)$ has mean dimension zero, then the C*-algebra $A_y$ is a locally approximately subhomogeneous C*-algebra with slow dimension growth.

**Proof.** This follows directly from Theorem 4.4. \hfill \Box

**Theorem 4.6.** Let $X$ be an infinite compact metrizable space, and let $\sigma : X \to X$ be a minimal homeomorphism. If $(X,\sigma)$ has mean dimension zero, then the C*-algebra $A = C(X) \rtimes_{\sigma} \mathbb{Z}$ absorbs the Jiang-Su algebra $\mathbb{Z}$ tensorially.

**Proof.** By Theorem 4.5, the C*-algebra $A_y$ is locally approximated by subhomogeneous C*-algebras with arbitrarily small dimension ratio. With Lemma 5.8, Lemma 5.10 of [11], the same argument as that of Theorem 1.2 of [15] shows that the Cuntz semigroups of $A_y$ and $A_y \otimes \mathbb{Z}$ are isomorphic, and therefore $A_y \cong A_y \otimes \mathbb{Z}$ by Corollary 7.4 of [18]. Since $A_y$ is centrally large in $A$ in the sense of D. Archey and N. C. Phillips, by [11], the nuclear C*-algebra $A$ also satisfies $A \cong A \otimes \mathbb{Z}$. \hfill \Box

**Corollary 4.7.** Let $(X,\sigma)$ be a minimal system with mean dimension zero, where $X$ is infinite. Consider $A = C(X) \rtimes_{\sigma} \mathbb{Z}$. Suppose that the projections of $A$ separate traces. Then $A$ belongs to the class of $\mathbb{Z}$-stable rationally AH algebras, and hence is classifiable. In particular, $A$ is classifiable if $(X,\sigma)$ is uniquely ergodic. (Note that, in this case, by Theorem 5.4 of [10], the hypothesis of mean dimension is redundant.)

**Proof.** Since $(X,\sigma)$ has mean dimension zero, the C*-algebra $A$ is Jiang-Su stable by Corollary 4.6. By [16] and [17], the C*-algebra $A$ is rationally AH. Therefore it is covered by the classification theorem of [19], [5], [6]. \hfill \Box

## 5. The Tensor Products

In this section, let us show that the tensor product of the crossed product C*-algebras of two or more minimal homeomorphisms is $\mathbb{Z}$-stable (Theorem 5.6). In particular, this implies that the Toms growth rank ([14]) of any crossed product C*-algebra $C(X) \rtimes_{\sigma} \mathbb{Z}$ with $(X,\sigma)$ minimal is at most two. This also shows that the examples of Giol and Kerr ([3]) are prime among the C*-algebras of minimal homeomorphisms.
Theorem 5.1. Let $X$ be an infinite compact metrizable space, and let $\sigma$ be a minimal homeomorphism. Let
\[ \{f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_m\} \subseteq C(X) \]
with $g_i(W) = \{0\}$, $i = 1, \ldots, m$, for some open set $W$ containing $y$. Then, for any $\varepsilon > 0$, there is $R > 0$ such that for any $J \in \mathbb{N}$, there is a closed neighborhood $Y$ of $y$ contained in $W$ such that the finite subset
\[ \{f_1, f_2, \ldots, f_n, u g_1, u g_2, \ldots, u g_m\} \]
of $A_Y$, where $u$ is the canonical unitary of the crossed product, is approximated to within $\varepsilon$ by a subhomogeneous $C^*$-algebra $S$ in $A_Y$ with dimension ratio at most $R$, and with the dimension of each irreducible representation at least $J$.

Proof. The proof is a slight modification of the proof of Theorem 4.4.

Let $\varepsilon > 0$ be arbitrary. Choose a finite open cover
\[ \alpha = \{U_1, U_2, \ldots, U_\alpha\} \]
of $X$ such that
\[ |f_i(x) - f_i(y)| < \varepsilon \quad \text{and} \quad |g_j(x) - g_j(y)| < \varepsilon, \quad x, y \in U_i, \quad 1 \leq i \leq |\alpha| . \]
Then
\[ R = |\alpha| + 1 \]
is the desired constant.

Let $J \in \mathbb{N}$ be arbitrary. Choose the closed neighborhood $Y$ of $y$ in $W$ as follows: the Rokhlin partition
\[ \{\{Z_1, \sigma(Z_1), \ldots, \sigma^{J_1-1}(Z_1)\}, \ldots, \{Z_k, \sigma(Z_k), \ldots, \sigma^{J_k-1}(Z_k)\}\} \]
corresponding as in Section 3 to $Y$ should satisfy
\[ J_1 \geq J. \]

Pick an open set $V$ such that $Y \subseteq V \subseteq W$, and pick a continuous function $H : X \to [0, 1]$ such that
\[ H^{-1}(0) = Y \quad \text{and} \quad H^{-1}(1) \supseteq (X \setminus V). \]
Since $Y \subseteq W$, without loss of generality, we may assume that $V \subseteq W$, and then
\[ H g_j = g_j, \quad 1 \leq j \leq m. \]
Choose a partition of unity $\{\phi_U : U \in \alpha\}$ subordinated to $\alpha$.

Let us show that the sub-$C^*$-algebra
\[ S := C^*\{\phi_U, u H ; \ U \in \alpha\} \subseteq A_Y, \]
together with the closed set $Y$, satisfies the conditions of the theorem (for $R$ and $J$).

For each $U \in \alpha$, pick a point $x_U \in U$. Then, by (5.1), for each $f_i$, $1 \leq i \leq n$, one has
\[ \left\| f_i - \sum_{U \in \alpha} f_i(x_U) \phi_U \right\| \leq \sup_{x \in X} \sum_{U \in \alpha} |f(x) - f_i(x_U)| \phi_U(x) < \varepsilon; \]
and for each $g_j$, $1 \leq j \leq m$, one has
\[
\left\| u g_j - u H \sum_{U \in \alpha} g_j(x_U) \phi_U \right\| = \left\| u H g_j - u H \sum_{U \in \alpha} g_j(x_U) \phi_U \right\| \\
\leq \left\| g_j - \sum_{U \in \alpha} g_j(x_U) \phi_U \right\| \\
< \varepsilon.
\]
This shows that
\[
\{ f_1, f_2, ..., f_n, u g_1, u g_2, ..., u g_m \} \subseteq \varepsilon S.
\]
Let us show that $\text{dimRatio}(S) \leq R$. For each $1 \leq k \leq K$, consider the algebra
\[
M_{J_k}(C(Z_k))
\]
of Theorem 3.2 and consider the map
\[
\xi_k : Z_k \rightarrow \mathbb{R}(|\alpha|+1)^{J_k-1}
\]
defined by
\[
(5.2) \quad \xi_k(x) \mapsto ((\Phi \circ \sigma(x), \Phi \circ \sigma^2(x), ..., \Phi \circ \sigma^{J_k}(x)), (H \circ \sigma(x), ..., H \circ \sigma^{J_k-1}(x))),
\]
where the map $\Phi : Z_k \rightarrow \mathbb{R}^{|\alpha|}$ is defined by
\[
\Phi = \bigoplus_{U \in \alpha} \phi_U.
\]
It is clear that image of $\xi_k(Z_k)$ has dimension at most $(|\alpha|+1)J_k - 1$. Then an argument similar to that of Theorem 4.4 shows that the irreducible representations of $S$ have dimension
\[
J_1 < J_2 < \cdots < J_K,
\]
and that
\[
\text{dim}(\text{Prim}_{J_k}(S)) \leq (|\alpha|+1)J_k - 1.
\]
Therefore,
\[
\text{dimRatio}(S) \leq |\alpha| + 1 = R,
\]
and the dimension of each irreducible representation of $S$ is at least $J$ (note that $J \leq J_1$). \hfill \Box

**Lemma 5.2.** Let $C$ and $S$ be subhomogeneous $C^*$-algebras, and let $m_0, n_0, m_1, n_1$, and $d$ be natural numbers satisfying $m_0 m_1 = n_0 n_1 = d$ and $m_0 \neq n_0$. Assume that $C$ has irreducible representations of dimensions $m_0$ and $n_0$, and that $S$ has irreducible representations of dimensions $m_1$ and $n_1$. Consider the subsets $E$ and $F$ of $X := \text{Prim}_d(C \otimes S)$ defined by
\[
E = \{ \rho : \rho = \pi_0 \otimes \pi_1, \ \pi_0 \in \text{Prim}_{m_0}(C), \ \pi_1 \in \text{Prim}_{m_1}(S) \}
\]
and
\[
F = \{ \rho : \rho = \pi_0 \otimes \pi_1, \ \pi_0 \in \text{Prim}_{n_0}(C), \ \pi_1 \in \text{Prim}_{n_1}(S) \}.
\]
Then the closures of $E$ and $F$ (in $X$) are disjoint. In particular, the sets $E$ and $F$ are relatively closed subsets of $X$. 


Proof. Assuming the contrary, there would be \((\pi_k^0 \otimes \pi_k^1)\) converging to \(\pi_\infty^0 \otimes \pi_\infty^1\) in \(\operatorname{Prim}_d(C \otimes S)\), where

\[
\pi_k^0 \in \operatorname{Prim}_{m_0}(C), \quad \pi_k^1 \in \operatorname{Prim}_{m_1}(S), \quad k = 1, 2, \ldots
\]

and

\[
\pi_\infty^0 \in \operatorname{Prim}_{m_\infty}(C), \quad \pi_\infty^1 \in \operatorname{Prim}_{n_\infty}(S), \quad k = 1, 2, \ldots.
\]

Without loss of generality, one may assume that \(m_0 < n_0\) (hence \(m_1 > n_1\)).

For any \(c \otimes s \in C \otimes S\), consider the sequence

\[
\operatorname{Tr}(\pi_k^0 \otimes \pi_k^1)(c \otimes s) = \operatorname{Tr}(\pi_k^0(c) \otimes \pi_k^1(s)) = \operatorname{Tr}(\pi_k^0(c)) \cdot \operatorname{Tr}(\pi_k^1(s)), \quad k = 1, 2, \ldots
\]

Since \((\pi_k^0 \otimes \pi_k^1) \to \pi_\infty^0 \otimes \pi_\infty^1\) in \(\operatorname{Prim}_d(C \otimes S)\), one has

\[
\operatorname{Tr}(\pi_k^0(c)) \cdot \operatorname{Tr}(\pi_k^1(s)) \to \operatorname{Tr}(\pi_\infty^0(c)) \cdot \operatorname{Tr}(\pi_\infty^1(s)), \quad k \to \infty.
\]

Setting \(s = 1_S\), one has that

\[
(5.3) \quad \operatorname{Tr}(\pi_k^0(c)) \to \frac{n_1}{m_1} \cdot \operatorname{Tr}(\pi_\infty^0(c)), \quad k \to \infty, \quad c \in C.
\]

Note that \((\pi_k^0) \subseteq \operatorname{Prim}_{m_0}(C), \pi_\infty^0 \in \operatorname{Prim}_{m_\infty}(C),\) and \(m_0 < n_0\). There is \(c \in C\) such that

\[
\pi_\infty^0(c) \neq 0 \quad \text{but} \quad \pi(c) = 0, \quad \pi \in \operatorname{Prim}_{m_\infty}(C).
\]

In particular,

\[
\pi_k^0(c) = 0, \quad k = 1, 2, \ldots.
\]

But this contradicts to \((5.3)\). \(\square\)

**Lemma 5.3.** Let \(C\) and \(S\) be unital subhomogeneous C*-algebras, and let \(J\) be a natural number such that each irreducible representation of \(C\) or \(S\) has dimension at least \(J\). Then

\[
\dim \operatorname{Ratio}(C \otimes S) \leq \frac{\dim \operatorname{Ratio}(C) + \dim \operatorname{Ratio}(S)}{J}.
\]

**Proof.** Let \(d\) be any natural number. Then

\[
\operatorname{Prim}_d(C \otimes S) = \bigcup_{m n = d} (\operatorname{Prim}_m(C) \times \operatorname{Prim}_n(S)).
\]

By Lemma 5.2 each \(\operatorname{Prim}_m(C) \times \operatorname{Prim}_n(S)\) is relatively close in \(\operatorname{Prim}_d(C \otimes S)\), and then one has

\[
\dim(\operatorname{Prim}_d(C \otimes S)) = \max_{m n = d} \{\dim(\operatorname{Prim}_m(C) \times \operatorname{Prim}_n(S))\}
\]

\[
\leq \max_{m n = d} \{\dim(\operatorname{Prim}_m(C)) + \dim(\operatorname{Prim}_n(S))\}.
\]

This implies

\[
\frac{\dim(\operatorname{Prim}_d(C \otimes S))}{d} \leq \max_{d = m n} \left\{\frac{\dim(\operatorname{Prim}_m(C)) + \dim(\operatorname{Prim}_n(S))}{d}\right\}
\]

\[
= \max_{d = m n} \left\{\frac{\dim(\operatorname{Prim}_m(C))}{m} \cdot \frac{1}{n} + \frac{\dim(\operatorname{Prim}_n(S))}{n} \cdot \frac{1}{m}\right\}
\]

\[
\leq \max_{d = m n} \{\dim \operatorname{Ratio}(C) \cdot \frac{1}{n} + \dim \operatorname{Ratio}(S) \cdot \frac{1}{m}\}
\]

\[
\leq \frac{\dim \operatorname{Ratio}(C) + \dim \operatorname{Ratio}(S)}{J},
\]

as desired. \(\square\)
Lemma 5.4. Let $A$ and $B$ be $C^*$-algebras satisfying the following: For any finite subset $F$ of $A$ (or $B$) and any $\varepsilon > 0$, there is $R > 0$ (which depends on $F$ and $\varepsilon$) and a sequence of unital sub-$C^*$-algebras $(S_n)$ such that

1. each $S_n$ is a subhomogeneous $C^*$-algebra with $\text{dimRatio}(S_n) \leq R$,
2. each $S_n$ approximately contains $F$ up to $\varepsilon$, and
3. $d_n \to \infty$ as $n \to \infty$, where $d_n$ is the smallest dimension of the irreducible representations of $S_n$.

Then $A \otimes B$ can be locally approximated by subhomogeneous $C^*$-algebras with slow dimension growth.

Proof. It is enough to show that for any finite subsets $F \subseteq A$, $G \subseteq B$, and any $\varepsilon \in (0, 1)$, there is a subhomogeneous $C^*$-algebra $D$ in $A \otimes B$ such that $F \otimes G \subseteq \varepsilon D$ and $\text{dimRatio}(D) < \varepsilon$.

Without loss of generality, one may assume that each element of $F$ and $G$ has norm one. By the assumptions, there are subhomogeneous $C^*$-algebras $C \subseteq A$ and $S \subseteq B$ such that $\text{dimRatio}(C) \leq R$ and $F \subseteq \varepsilon^{-1} C$, and $\text{dimRatio}(S) \leq R$ and $G \subseteq \varepsilon^{-1} S$, and, furthermore, the dimension of each irreducible representation of $C$ or $S$ is at least $\frac{2R}{\varepsilon}$. Then consider the $C^*$-algebra

$$D := C \otimes S.$$ 

By Lemma 5.3 one has

$$\text{dimRatio}(D) \leq \varepsilon.$$ 

A straightforward calculation also shows that

$$F \otimes G \subseteq \varepsilon D,$$

and this finishes the proof. \hfill \Box

Proposition 5.5. Let $(X_1, \sigma_1)$ and $(X_2, \sigma_2)$ be minimal systems, where $X_1$ and $X_2$ are infinite. Fix $y_1 \in X_1$ and $y_2 \in X_2$, and consider the large sub-$C^*$-algebras

$$A_{y_1} \subseteq C(X_1) \rtimes_{\sigma_1} \mathbb{Z} \quad \text{and} \quad A_{y_2} \subseteq C(X_2) \rtimes_{\sigma_2} \mathbb{Z}.$$ 

Then

$$A_{y_1} \otimes A_{y_2} \cong (A_{y_1} \otimes A_{y_2}) \otimes \mathbb{Z}.$$ 

Proof. By Theorem 5.1 and Lemma 5.4, the $C^*$-algebra $A_{y_1} \otimes A_{y_2}$ is locally approximated by subhomogeneous $C^*$-algebras with slow dimension growth, and therefore it absorbs the Jiang-Su algebra tensorially. \hfill \Box

Theorem 5.6. Let $(X_1, \sigma_1)$ and $(X_2, \sigma_2)$ be minimal systems, where $X_1$ and $X_2$ are infinite compact metrizable spaces. Consider the $C^*$-algebras

$$A_1 = C(X_1) \rtimes_{\sigma_1} \mathbb{Z} \quad \text{and} \quad A_2 = C(X_2) \rtimes_{\sigma_2} \mathbb{Z}.$$ 

Then

$$A_1 \otimes A_2 \cong (A_1 \otimes A_2) \otimes \mathbb{Z}.$$ 

In particular, the crossed product $C^*$-algebra of a minimal homeomorphism has Toms growth rank (\cite{14}) at most 2.
Proof. By Proposition 5.5, the C*-algebra $A_{y_1} \otimes A_{y_2}$ absorbs the Jiang-Su algebra $\mathcal{Z}$. By Lemma 3.5, $A_{y_1} \otimes A_{y_2}$ is centrally large in $A_1 \otimes A_2$. By [13], the nuclear C*-algebra $A_1 \otimes A_2$ absorbs the Jiang-Su algebra. □

Remark 5.7. Note that $A_1 \otimes A_2 \cong C(X_1 \times X_2) \rtimes_{\sigma} \mathbb{Z}^2$, where the action of $\mathbb{Z}^2$ on $X_1 \times X_2$ is given by $\sigma_{(m,n)}(x_1, x_2) = (\sigma_1^m(x_1), \sigma_2^n(x_2))$.

In particular, Theorem 5.6 implies that for minimal actions of $\mathbb{Z}$ on $X_1$ and $X_2$ the crossed product C*-algebra $C(X_1 \times X_2) \rtimes_{\sigma} \mathbb{Z}^2$ always absorbs the Jiang-Su algebra.

References


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