Basic Homotopy Lemmas
Introduction

Huaxin Lin

June 8th, 2015, RMMC/CBMS University of Wyoming
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We need to assume that $v \in U_0(A)$, the connected component of unitary group $U(A)$ containing $1_A$.

If the answer is yes, how long is the length of the path?
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\|uv - vu\| < \delta,
\]

then there exists a continuous path of unitaries \( \{v(t) : t \in [0, 1]\} \subset A \) such that

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\( v(0) = v \) and \( v(1) = 1_A \)?

We need to assume that \( v \in U_0(A) \), the connected component of unitary group \( U(A) \) containing \( 1_A \). If the answer is yes, how long is the length of the path?
Theorem  (The Basic Homotopy Lemma—Bratteli, Elliott, Evans and Kishimoto–1998)

Let $\epsilon > 0$. There exists $\delta > 0$ satisfying the following:

For any unital simple $C^*$-algebra $A$ of stable rank one and real rank zero and any pair of unitaries $u, v \in A$ with $u \in U_0(A)$ such that $\|uv - vu\| < \delta$ and $bott_1(u, v) = 0$, (e 0.1) there exists a continuous path of unitaries $\{u(t) : t \in [0, 1]\} \subset A$ such that $u(0) = u, u(1) = 1_A$ and $\|u(t)v - vu(t)\| < \epsilon$ for all $t \in [0, 1]$ and (e 0.2) length $\{u(t)\} \leq 4\pi + 1$. (e 0.3)
**Theorem** *(The Basic Homotopy Lemma—Bratteli, Elliott, Evans and Kishimoto–1998)*

Let $\epsilon > 0$. 

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<tr>
<th>$\mathbf{0.1}$</th>
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$\text{length}(\{u(t)\}) \leq 4\pi + 1.$
Lemma 1.1.

Let $\epsilon > 0$ and let $d > 0$, there exists $\delta > 0$ satisfying the following: Suppose that $A$ is a unital $C^*$-algebra and $u \in A$ is a unitary such that $T \setminus \text{sp}(u)$ contains an arc with length $d$. Suppose that $a \in A$ with $\|a\| \leq 1$ such that $\|ua - au\| < \delta$. Then there exists a self-adjoint element $h \in A$ with $\|h\| \leq \pi$ such that $u = \exp(ith)$, $\|ha - ah\| < \epsilon$ and $\|\exp(i\theta h)a - a\exp(i\theta h)\| < \epsilon$ (e 0.5) for all $t \in [0, 1]$. 
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sp(u) \subset \Omega_d = \{ e^{i\pi t} : -1 + d/2 \leq t \leq 1 - d/2 \} \subset \mathbb{T}.
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There is a continuous function $g : \Omega_d \to [-1, 1]$ such that $u = \exp(ig(u))$. 
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There is a continuous function $g : \Omega_d \to [-1, 1]$ such that $u = \exp(ig(u))$. Let $h = g(u)$. Choose an integer $N \geq 1$ such that

$$\sum_{i=N+1}^{\infty} 1/n! < \epsilon/6.$$  \hspace{1cm} (e 0.7)

There is $\delta > 0$ such that $\|au - ua\| < \delta$ implies that $\|h_n a - ah_n\| = \|g(u_n) a - ag(u_n)\| < \epsilon/6$ for $n = 1, 2, \ldots, N$.\hspace{1cm}
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implies that \( \|h^n a - ah^n\| = \|g(u)^n a - ag(u)^n\| < \epsilon/6 \) for \( n = 1, 2, \ldots, N \).
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\[ \| \exp(i \theta) a - a \exp(i \theta) \| \leq N \sum_{n=0}^{\infty} \frac{|i \theta|^n}{n!} + 2N \sum_{n=N+1}^{\infty} \frac{1}{n!} e^{0.1} \leq \epsilon_6 n! + \epsilon/3 < \epsilon. \] (e 0.9)
Then

\[
\| \exp(ith)a - a \exp(ith) \| \\
\leq \| \left( \sum_{n=0}^{N} \frac{ith^n}{n!} \right) a - a \left( \sum_{n=0}^{N} \frac{ith^n}{n!} \right) \| + 2 \left( \sum_{n=N+1}^{\infty} \frac{1}{n!} \right)
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Then

\[ \| \exp(ith) a - a \exp(ith) \| \leq \left\| \left( \sum_{n=0}^{N} \frac{ith^n}{n!} \right) a - a \left( \sum_{n=0}^{N} \frac{ith^n}{n!} \right) \right\| + 2 \left( \sum_{n=N+1}^{\infty} \frac{1}{n!} \right) \]

\[ \leq \sum_{n=1}^{N} \frac{\epsilon}{6n!} + \epsilon/3 < \epsilon. \]

for any \( t \in [0, 1] \).
Corollary 1.2.

Let $n \geq 1$ be an integer.
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Let $n \geq 1$ be an integer. Let $C$ be a unital $C^*$-algebra and let $\mathcal{F} \subset C$ be a finite subset. For any $\epsilon > 0$ there exists $\delta > 0$ satisfying the following: Suppose $L : C \to M_n$ is a contractive map.

Proof. The spectrum of $u$ has a gap with the length at least $d = \frac{2\pi}{n}$. 

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Corollary 1.2.

Let $n \geq 1$ be an integer. Let $C$ be a unital $C^*$-algebra and let $\mathcal{F} \subset C$ be a finite subset. For any $\epsilon > 0$ there exists $\delta > 0$ satisfying the following:

Suppose $L : C \to M_n$ is a contractive map and $u \in M_n$ is a unitary such that

$$\|L(c)u - uL(c)\| < \delta \text{ for all } c \in \mathcal{F}. \quad (e\ 0.12)$$

Moreover, length $(u(t)) \leq \pi$. 

Proof. The spectrum of $u$ has a gap with the length at least $d = \frac{2\pi}{n}$.
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Then there exists a continuous path of unitaries $\{u(t) : t \in [0, 1]\} \subset M_n$ such that

$$\|L(c)u(t) - u(t)L(c)\| < \epsilon \text{ for all } c \in \mathcal{F}.$$  

Moreover, the length of $u(t)$ is at most $\pi$. 

Proof. The spectrum of $u$ has a gap with the length at least $\frac{2\pi}{n}$. 

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\]

Moreover, \( \text{length}(u(t)) \leq \pi \).

Proof.
The spectrum of \( u \) has a gap with the length at least \( d = 2\pi/n \).\qed
Let $R \subset \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ be a subset and let $A \subset \{1, 2, \ldots, m\}$.
Let $R \subset \{1, 2, ..., m\} \times \{1, 2, ..., n\}$ be a subset and let $A \subset \{1, 2, ..., m\}$. Define $R_A \subset \{1, 2, ..., n\}$ to be the subset of those $j$'s such that $(i, j) \in R$, for some $i \in A$. The following follows from Hall's Marriage lemma.

Lemma 1.3. If $\{a_i\}_{i=1}^{m}$, $\{b_j\}_{j=1}^{n} \subset \mathbb{Z}_k^+$ with $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, and $R \subset \{1, 2, ..., m\} \times \{1, 2, ..., n\}$ satisfying:

$$\sum_{i \in A} a_i \leq \sum_{j \in R_A} b_j,$$

then there are $\{c_{ij}\} \subset \mathbb{Z}_k^+$ such that $\sum_{j=1}^{n} c_{ij} = a_i$, $\sum_{i=1}^{m} c_{ij} = b_j$, for all $i, j$. (e 0.15)

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Let $R \subset \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ be a subset and let $A \subset \{1, 2, \ldots, m\}$. Define $R_A \subset \{1, 2, \ldots, n\}$ to be the subset of those $j'$s such that $(i, j) \in R$, for some $i \in A$. The following follows from Hall’s Marriage lemma.
Let $R \subset \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ be a subset and let $A \subset \{1, 2, \ldots, m\}$. Define $R_A \subset \{1, 2, \ldots, n\}$ to be the subset of those $j$’s such that $(i, j) \in R$, for some $i \in A$. The following follows from Hall’s Marriage lemma.

**Lemma 1.3.**

If $\{a_i\}_{i=1}^m, \{b_i\}_{j=1}^n \subset \mathbb{Z}_+$ then there are $\{c_{ij}\} \subset \mathbb{Z}_+$ such that

$$
\sum_{j=1}^n c_{ij} = a_i, \quad \sum_{i=1}^m c_{ij} = b_j,
$$

for all $i, j$, and $c_{ij} = 0$ unless $(i, j) \in R$. 

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Huaxin Lin

Basic Homotopy Lemmas Introduction

June 8th, 2015, RMMC/CBMS University of Wyoming
Let $R \subset \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ be a subset and let $A \subset \{1, 2, \ldots, m\}$. Define $R_A \subset \{1, 2, \ldots, n\}$ to be the subset of those $j$'s such that $(i, j) \in R$, for some $i \in A$. The following follows from Hall’s Marriage lemma.

**Lemma 1.3.**

If $\{a_i\}_{i=1}^{m}, \{b_j\}_{j=1}^{n} \subset \mathbb{Z}_+^{k}$ with $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, then there are $\{c_{ij}\} \subset \mathbb{Z}_+^{k}$ such that

$$\sum_{j=1}^{n} c_{ij} = a_i, \quad \sum_{i=1}^{m} c_{ij} = b_j,$$

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If \( \{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n \subset \mathbb{Z}_+^k \) with \( \sum_{i=1}^m a_i = \sum_{j=1}^n b_j \), and \( R \subset \{1, \ldots, m\} \times \{1, \ldots, n\} \) satisfying: for any \( A \subset \{1, \ldots, m\} \),

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\]

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\[
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Lemma 1.4.

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Let $X$ be a (connected) compact metric space, let $P \in M_r(C(X))$ be a projection and let $n \geq 1$ be an integer. Let $\epsilon > 0$ and let $\mathcal{F} \subset C = PM_r(C(X))P$ be a finite subset.

There exists $\delta > 0$ and a finite subset $H \subset C$ satisfying the following. Suppose that $\phi, \psi : C \to M_n$ are two unital homomorphisms such that $|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta$ for all $g \in H$ (e 0.17) ($\tau$ is the tracial state on $M_n$). Then there exists a unitary $u \in U(M_n)$ such that $\|Ad_u \circ \phi(a) - \psi(a)\| < \epsilon$ for all $a \in \mathcal{F}$ (e 0.18).

Moreover, if $\phi(a) = \sum_{i=1}^n f(x_i) e_i$ and $\psi(a) = \sum_{j=1}^n f(y_j) e'_j$ for all $f \in C$, where $x_i, y_j \in X$, $\{e_1, e_2, \ldots, e_n\}$ and $\{e'_1, e'_2, \ldots, e'_n\}$ are two sets of mutually orthogonal projections, and if $d > 0$ is given, then we may assume that there is a permutation $\sigma$ on $\{1, 2, \ldots, n\}$ such that $u^* e_i u = e'_{\sigma(i)}$, and $\operatorname{dist}(x_i, y_{\sigma(i)}) < d$, $i = 1, 2, \ldots, n$, where $\delta$ depends on $\epsilon, F$ and $d$. 
Lemma 1.4.

Let $X$ be a (connected) compact metric space, let $P \in M_r(C(X))$ be a projection and let $n \geq 1$ be an integer. Let $\epsilon > 0$ and let $\mathcal{F} \subset C = PM_r(C(X))P$ be a finite subset. There exists $\delta > 0$ and a finite subset $\mathcal{H} \subset C_{s.a.}$ satisfying the following.
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$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \quad \text{for all } g \in \mathcal{H}$$

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Huaxin Lin
Proof:
We will prove the case that $C = C(X)$. 

There exists $\eta > 0$ such that

$$\|f(x) - f(y)\| < \epsilon/4$$

for all $f \in F$, provided that $\text{dist}(x, y) < \eta$.

Let $O_1, O_2, \ldots, O_m$ be a finite open cover such that each $O_i$ has diameter $< \eta/4$.

Let $J \subset \{1, 2, \ldots, m\}$ be a subset.

Define $g_J \in C(X)$ such that $0 \leq g_J \leq 1$, $g_J(x) = 1$ if $x \in \bigcup_{i \in J} O_i$, $g_J(x) = 0$ if $\text{dist}(x, \bigcup_{i \in J} O_i) \geq \eta/4$.

Let $h_J \in C(X)$ be such that $0 \leq h_J \leq 1$, $h_J(x) = 1$ if $\text{dist}(x, \bigcup_{i \in J} O_i) < \eta/2$ and $h_J(x) = 0$ if $\text{dist}(x, \bigcup_{i \in J} O_i) > \eta$.

Let $\delta = \min\{\eta/16, 1/16\}$.

Set $H = \{g_I, h_J : I, J \subset \{1, 2, \ldots, m\}\}$.

Now suppose that $\phi, \psi : C(X) \to \mathbb{M}_n$ such that $|\tau \circ \phi(c) - \tau \circ \psi(c)| < \delta$ for all $c \in H$.

We have, for all $f \in C(X)$,

$$\phi(f) = k_1 \sum_{i=1} f(x_i) p_i \quad \text{and} \quad \psi(f) = k_2 \sum_{i=1} f(y_i) q_i.$$
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Proof:

We will prove the case that $C = C(X)$. There exists $\eta > 0$ such that
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\|f(x) - f(y)\| < \epsilon/4 \text{ for all } f \in \mathcal{F},
\]
provided that $\text{dist}(x, y) < \eta$. Let $O_1, O_2, ..., O_m$ be a finite open cover such that each $O_i$ has diameter $< \eta/4$. Let $J \subset \{1, 2, ..., m\}$ be a subset. Define $g_J \in C(X)_\pm$ such that $0 \leq g_J \leq 1$, $g_J(x) = 1$ if $x \in \bigcup_{i \in J} O_i$, $g_J(x) = 0$ if $\text{dist}(x, \bigcup_{i \in J} O_i) \geq \eta/4$. Let $h_J \in C(X)$ be such that $0 \leq h_J \leq 1$, $h_J(x) = 1$ if $\text{dist}(x, \bigcup_{i \in J} O_i) < \eta/2$ and $h_J(x) = 0$ if $\text{dist}(x, \bigcup_{i \in J} O_i) > \eta$. Let $\delta = \min\{\eta/16n, 1/16n\}$. Set
\[
\mathcal{H} = \{g_I, h_J : I, J \subset \{1, 2, ..., m\}\}.
\]
Proof:

We will prove the case that $C = C(X)$. There exists $\eta > 0$ such that

$$\|f(x) - f(y)\| < \epsilon/4 \text{ for all } f \in F,$$  \hfill (e 0.19)

provided that $\text{dist}(x, y) < \eta$. Let $O_1, O_2, ..., O_m$ be a finite open cover such that each $O_i$ has diameter $< \eta/4$. Let $J \subset \{1, 2, ..., m\}$ be a subset. Define $g_J \in C(X)_+$ such that $0 \leq g_J \leq 1$, $g_J(x) = 1$ if $x \in \bigcup_{i \in J} O_i$, $g_J(x) = 0$ if $\text{dist}(x, \bigcup_{i \in J} O_i) \geq \eta/4$. Let $h_J \in C(X)$ be such that $0 \leq h_J \leq 1$, $h_J(x) = 1$ if $\text{dist}(x, \bigcup_{i \in J} O_i) < \eta/2$ and $h_J(x) = 0$ if $\text{dist}(x, \bigcup_{i \in J} O_i) > \eta$. Let $\delta = \min\{\eta/16n, 1/16n\}$. Set

$$\mathcal{H} = \{g_I, h_J : I, J \subset \{1, 2, ..., m\}\}. \hfill (e 0.20)$$

Now suppose that $\phi, \psi : C(X) \to M_n$ such that

$$|\tau \circ \phi(c) - \tau \circ \psi(c)| < \delta \text{ for all } c \in \mathcal{H}. \hfill (e 0.21)$$
Proof:
We will prove the case that $C = C(X)$. There exists $\eta > 0$ such that
\[ \| f(x) - f(y) \| < \epsilon/4 \text{ for all } f \in \mathcal{F}, \] (e 0.19)
provided that $\text{dist}(x, y) < \eta$. Let $O_1, O_2, \ldots, O_m$ be a finite open cover such that each $O_i$ has diameter $< \eta/4$. Let $J \subset \{1, 2, \ldots, m\}$ be a subset. Define $g_J \in C(X)_+$ such that $0 \leq g_J \leq 1$, $g_J(x) = 1$ if $x \in \bigcup_{i \in J} O_i$, $g_J(x) = 0$ if $\text{dist}(x, \bigcup_{i \in J} O_i) \geq \eta/4$. Let $h_J \in C(X)$ be such that $0 \leq h_J \leq 1$, $h_J(x) = 1$ if $\text{dist}(x, \bigcup_{i \in J} O_i) < \eta/2$ and $h_J(x) = 0$ if $\text{dist}(x, \bigcup_{i \in J} O_i) > \eta$. Let $\delta = \min\{\eta/16n, 1/16n\}$. Set
\[ \mathcal{H} = \{g_I, h_J : I, J \subset \{1, 2, \ldots, m\}\}. \] (e 0.20)
Now suppose that $\phi, \psi : C(X) \to M_n$ such that
\[ |\tau \circ \phi(c) - \tau \circ \psi(c)| < \delta \text{ for all } c \in \mathcal{H}. \] (e 0.21)
We have, for all $f \in C(X)$,
\[ \phi(f) = \sum_{i=1}^{k_1} f(x_i)p_i \text{ and } \psi(f) = \sum_{i=1}^{k_2} f(y_i)q_i, \] (e 0.22)
where $X_0 = \{x_1, x_2, \ldots, x_{k_1}\}$ and $Y_0 = \{y_1, y_2, \ldots, y_{k_2}\}$ are finite subsets of $X$.
where $X_0 = \{x_1, x_2, \ldots, x_{k_1}\}$ and $Y_0 = \{y_1, y_2, \ldots, y_{k_2}\}$ are finite subsets of $X$, and $\{p_1, p_2, \ldots, p_{k_1}\}$ and $\{q_1, q_2, \ldots, q_{k_2}\}$ are two sets of mutually orthogonal projections such that $\sum_{i=1}^{k_1} p_i = \sum_{j=1}^{k_2} q_j = 1_{M_n}$.
where \( X_0 = \{ x_1, x_2, \ldots, x_{k_1} \} \) and \( Y_0 = \{ y_1, y_2, \ldots, y_{k_2} \} \) are finite subsets of \( X \), and \( \{ p_1, p_2, \ldots, p_{k_1} \} \) and \( \{ q_1, q_2, \ldots, q_{k_2} \} \) are two sets of mutually orthogonal projections such that \( \sum_{i=1}^{k_1} p_i = \sum_{j=1}^{k_2} q_j = 1_{M_n} \). Define a subset \( R \subset \{ 1, 2, \ldots, k_1 \} \times \{ 1, 2, \ldots, k_2 \} \) as follows:
where $X_0 = \{x_1, x_2, \ldots, x_{k_1}\}$ and $Y_0 = \{y_1, y_2, \ldots, y_{k_2}\}$ are finite subsets of $X$, and $\{p_1, p_2, \ldots, p_{k_1}\}$ and $\{q_1, q_2, \ldots, q_{k_2}\}$ are two sets of mutually orthogonal projections such that $\sum_{i=1}^{k_1} p_i = \sum_{j=1}^{k_2} q_j = 1_{M_n}$. Define a subset $R \subset \{1, 2, \ldots, k_1\} \times \{1, 2, \ldots, k_2\}$ as follows: $(i, j) \in R$ if and only if $\operatorname{dist}(x_i, y_j) < \eta$. 

Let $\phi$ be the range projection of $\psi(gA)$ in $M_n$. Define $\tau(\phi(gA)) \geq \sum_{x_i \in S} a_i / n$ (e 0.23).

Then $\tau(\psi(gA)) \geq \sum_{x_i \in S} a_i / n - 1 / 16 n$ (e 0.24).

Therefore $\tau(\psi(h)) \geq \sum_{x_i \in S} a_i / n$. (e 0.26)
where \( X_0 = \{ x_1, x_2, \ldots, x_{k_1} \} \) and \( Y_0 = \{ y_1, y_2, \ldots, y_{k_2} \} \) are finite subsets of \( X \), and \( \{ p_1, p_2, \ldots, p_{k_1} \} \) and \( \{ q_1, q_2, \ldots, q_{k_2} \} \) are two sets of mutually orthogonal projections such that \( \sum_{i=1}^{k_1} p_i = \sum_{j=1}^{k_2} q_j = 1_{M_n} \). Define a subset \( R \subset \{ 1, 2, \ldots, k_1 \} \times \{ 1, 2, \ldots, k_2 \} \) as follows: \( (i, j) \in R \) if and only if \( \text{dist}(x_i, y_j) < \eta \). Let \( a_i = \text{rank} p_i \) and \( b_j = \text{rank} q_j \).
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where $X_0 = \{x_1, x_2, \ldots, x_{k_1}\}$ and $Y_0 = \{y_1, y_2, \ldots, y_{k_2}\}$ are finite subsets of $X$, and $\{p_1, p_2, \ldots, p_{k_1}\}$ and $\{q_1, q_2, \ldots, q_{k_2}\}$ are two sets of mutually orthogonal projections such that $\sum_{i=1}^{k_1} p_i = \sum_{j=1}^{k_2} q_j = 1_{M_n}$. Define a subset $R \subset \{1, 2, \ldots, k_1\} \times \{1, 2, \ldots, k_2\}$ as follows: $(i, j) \in R$ if and only if $\text{dist}(x_i, y_j) < \eta$. Let $a_i = \text{rank}p_i$ and $b_j = \text{rank}q_j$. Let $S \subset X_0$ be a subset. Put $A = \{i \in \{1, 2, \ldots, k_1\} : x_i \in S\}$. Then

$$
\tau(\phi(g_A)) \geq \sum_{x_i \in S} a_i / n. \quad (e\ 0.23)
$$

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where $X_0 = \{x_1, x_2, \ldots, x_{k_1}\}$ and $Y_0 = \{y_1, y_2, \ldots, y_{k_2}\}$ are finite subsets of $X$, and \{p_1, p_2, \ldots, p_{k_1}\} and \{q_1, q_2, \ldots, q_{k_2}\} are two sets of mutually orthogonal projections such that \(\sum_{i=1}^{k_1} p_i = \sum_{j=1}^{k_2} q_j = 1_{M_n}\). Define a subset $R \subset \{1, 2, \ldots, k_1\} \times \{1, 2, \ldots, k_2\}$ as follows: $(i, j) \in R$ if and only if $\text{dist}(x_i, y_j) < \eta$. Let $a_i = \text{rank} p_i$ and $b_j = \text{rank} q_j$. Let $S \subset X_0$ be a subset. Put $A = \{i \in \{1, 2, \ldots, k_1\} : x_i \in S\}$. Then

$$\tau(\phi(g_A)) \geq \sum_{x_i \in S} a_i/n.$$  \hfill (e 0.23)

It follows that

$$\tau(\psi(g_A)) \geq \sum_{x_i \in S} a_i/n - 1/16n.$$  \hfill (e 0.24)
where $X_0 = \{x_1, x_2, ..., x_{k_1}\}$ and $Y_0 = \{y_1, y_2, ..., y_{k_2}\}$ are finite subsets of $X$, and \{\(p_1, p_2, ..., p_{k_1}\)\} and \{\(q_1, q_2, ..., q_{k_2}\)\} are two sets of mutually orthogonal projections such that $\sum_{i=1}^{k_1} p_i = \sum_{j=1}^{k_2} q_j = 1_{M_n}$. Define a subset $R \subset \{1, 2, ..., k_1\} \times \{1, 2, ..., k_2\}$ as follows: $(i, j) \in R$ if and only if $\text{dist}(x_i, y_j) < \eta$. Let $a_i = \text{rank} p_i$ and $b_j = \text{rank} q_j$. Let $S \subset X_0$ be a subset. Put $A = \{i \in \{1, 2, ..., k_1\} : x_i \in S\}$. Then

$$\tau(\phi(g_A)) \geq \sum_{x_i \in S} a_i / n. \quad \text{(e 0.23)}$$

It follows that

$$\tau(\psi(g_A)) \geq \sum_{x_i \in S} a_i / n - 1 / 16n. \quad \text{(e 0.24)}$$

Let $P_S$ be the range projection of $\psi(g_A)$ in $M_n$,
where $X_0 = \{x_1, x_2, ..., x_{k_1}\}$ and $Y_0 = \{y_1, y_2, ..., y_{k_2}\}$ are finite subsets of $X$, and $\{p_1, p_2, ..., p_{k_1}\}$ and $\{q_1, q_2, ..., q_{k_2}\}$ are two sets of mutually orthogonal projections such that $\sum_{i=1}^{k_1} p_i = \sum_{j=1}^{k_2} q_j = 1_{M_n}$. Define a subset $R \subset \{1, 2, ..., k_1\} \times \{1, 2, ..., k_2\}$ as follows: $(i, j) \in R$ if and only if $\text{dist}(x_i, y_j) < \eta$. Let $a_i = \text{rank} p_i$ and $b_j = \text{rank} q_j$. Let $S \subset X_0$ be a subset. Put $A = \{i \in \{1, 2, ..., k_1\} : x_i \in S\}$. Then

$$\tau(\phi(g_A)) \geq \sum_{x_i \in S} a_i/n. \quad (e 0.23)$$

It follows that

$$\tau(\psi(g_A)) \geq \sum_{x_i \in S} a_i/n - 1/16n. \quad (e 0.24)$$

Let $P_S$ be the range projection of $\psi(g_A)$ in $M_n$. Then

$$\tau(P_S) \geq \sum_{x_i \in S} a_i/n = \sum_{i \in A} a_i/n. \quad (e 0.25)$$
where \( X_0 = \{x_1, x_2, ..., x_{k_1}\} \) and \( Y_0 = \{y_1, y_2, ..., y_{k_2}\} \) are finite subsets of \( X \), and \( \{p_1, p_2, ..., p_{k_1}\} \) and \( \{q_1, q_2, ..., q_{k_2}\} \) are two sets of mutually orthogonal projections such that \( \sum_{i=1}^{k_1} p_i = \sum_{j=1}^{k_2} q_j = 1_{M_n} \). Define a subset \( R \subset \{1, 2, ..., k_1\} \times \{1, 2, ..., k_2\} \) as follows: \( (i, j) \in R \) if and only if \( \text{dist}(x_i, y_j) < \eta \). Let \( a_i = \text{rank} p_i \) and \( b_j = \text{rank} q_j \). Let \( S \subset X_0 \) be a subset. Put \( A = \{i \in \{1, 2, ..., k_1\} : x_i \in S\} \). Then

\[
\tau(\phi(g_A)) \geq \sum_{x_i \in S} a_i / n. \tag{e 0.23}
\]

It follows that

\[
\tau(\psi(g_A)) \geq \sum_{x_i \in S} a_i / n - 1/16n. \tag{e 0.24}
\]

Let \( P_S \) be the range projection of \( \psi(g_A) \) in \( M_n \). Then

\[
\tau(P_S) \geq \sum_{x_i \in S} a_i / n = \sum_{i \in A} a_i / n. \tag{e 0.25}
\]

Therefore

\[
\tau(\psi(h_i)) \geq \sum_{i \in A} a_i / n. \tag{e 0.26}
\]
It follows that

$$\sum_{i \in A} a_i \leq \sum_{j \in R_A} b_j.$$  \hfill (e 0.27)
It follows that

\[
\sum_{i \in A} a_i \leq \sum_{j \in R_A} b_j. \tag{e 0.27}
\]

This holds, for any subset \( A \subset \{1, 2, \ldots, k_1\} \).
It follows that

$$\sum_{i \in A} a_i \leq \sum_{j \in R_A} b_j. \quad (e\ 0.27)$$

This holds, for any subset $A \subset \{1, 2, \ldots, k_1\}$. By the previous lemma 1.3, there are $\{c_{i,j}\} \subset \mathbb{Z}^+$ such that

$$\sum_{j=1}^{k_2} c_{ij} = a_i, \quad \sum_{i=1}^{k_1} c_{ij} = b_j \quad (e\ 0.28)$$

and $c_{ij} \neq 0$ if and only if $(i, j) \in R$. 

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It follows that

\[ \sum_{i \in A} a_i \leq \sum_{j \in R_A} b_j. \quad (e \, 0.27) \]

This holds, for any subset \( A \subset \{1, 2, \ldots, k_1\} \). By the previous lemma 1.3, there are \( \{c_{i,j}\} \subset \mathbb{Z}^+ \) such that

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It follows that
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\[
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\]
and \( c_{ij} \neq 0 \) if and only \((i, j) \in R\). Therefore there are mutually orthogonal projections \( p_{ij} \) and \( q_{ij} \) such that
\[
\sum_{j=1}^{k_2} p_{ij} = p_i, \quad \sum_{i=1}^{k_1} q_{ij} = q_j \quad (e \, 0.29)
\]
It follows that

\[
\sum_{i \in A} a_i \leq \sum_{j \in R_A} b_j. \tag{e 0.27}
\]

This holds, for any subset \( A \subset \{1, 2, \ldots, k_1\} \). By the previous lemma \( \textbf{1.3} \), there are \( \{c_{i,j}\} \subset \mathbb{Z}^+ \) such that

\[
\sum_{j=1}^{k_2} c_{ij} = a_i, \quad \sum_{i=1}^{k_1} c_{ij} = b_j \tag{e 0.28}
\]

and \( c_{ij} \neq 0 \) if and only \((i, j) \in R\). Therefore there are mutually orthogonal projections \( p_{ij} \) and \( q_{ij} \) such that

\[
\sum_{j=1}^{k_2} p_{ij} = p_i, \quad \sum_{i=1}^{k_1} q_{ij} = q_j \tag{e 0.29}
\]

\[\text{rank} p_{ij} = \text{rank} q_{ij}\]
It follows that

\[ \sum_{i \in A} a_i \leq \sum_{j \in R_A} b_j. \]  

(e 0.27)

This holds, for any subset \( A \subset \{1, 2, \ldots, k_1\} \). By the previous lemma 1.3, there are \( \{c_{i,j}\} \subset \mathbb{Z}^+ \) such that

\[ \sum_{j=1}^{k_2} c_{ij} = a_i, \quad \sum_{i=1}^{k_1} c_{ij} = b_j \]  

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\[ \sum_{j=1}^{k_2} p_{ij} = p_i, \quad \sum_{i=1}^{k_1} q_{ij} = q_j \]  

(e 0.29)

\[ \text{rank} p_{ij} = \text{rank} q_{ij} \quad \text{and} \quad p_{ij} \neq 0 \quad \text{and} \quad q_{ij} \neq 0 \text{ if and only if } (i, j) \in R. \]
It follows that
\[ \sum_{i \in A} a_i \leq \sum_{j \in R_A} b_j. \] (e 0.27)

This holds, for any subset \( A \subset \{1, 2, \ldots, k_1\} \). By the previous lemma 1.3, there are \( \{c_{i,j}\} \subset \mathbb{Z}^+ \) such that
\[ \sum_{j=1}^{k_2} c_{ij} = a_i, \quad \sum_{i=1}^{k_1} c_{ij} = b_j \] (e 0.28)

and \( c_{ij} \neq 0 \) if and only \((i, j) \in R\). Therefore there are mutually orthogonal projections \( p_{ij} \) and \( q_{ij} \) such that
\[ \sum_{j=1}^{k_2} p_{ij} = p_i, \quad \sum_{i=1}^{k_1} q_{ij} = q_j \] (e 0.29)

\( \text{rank} p_{ij} = \text{rank} q_{ij} \) and \( p_{ij} \neq 0 \) and \( q_{ij} \neq 0 \) if and only if \((i, j) \in R\).
We may write

\[ \phi(f) = \sum_{i,j} f(x_i) p_{ij} \quad \text{and} \quad \psi(f) = \sum_{i,j} f(y_j) q_{ij}. \]  

(e 0.30)
We may write

\[ \phi(f) = \sum_{i,j} f(x_i)p_{ij} \quad \text{and} \quad \psi(f) = \sum_{i,j} f(y_j)q_{ij}. \]  

\[(e\ 0.30)\]

Moreover, \( p_{ij} \neq 0 \) and \( q_{ij} \neq 0 \) if and only if \( \text{dist}(x_i, y_j) < \eta \).
We may write

\[ \phi(f) = \sum_{i,j} f(x_i)p_{ij} \quad \text{and} \quad \psi(f) = \sum_{i,j} f(y_j)q_{ij}. \]  

(e 0.30)

Moreover, \( p_{ij} \neq 0 \) and \( q_{ij} \neq 0 \) if and only if \( \text{dist}(x_i, y_j) < \eta \). Therefore there exists a unitary \( u \in M_n \) such that

\[ u^* p_{ij} u = q_{ij} \quad \text{and} \quad \| \text{Ad} u \circ \phi(f) - \psi(f) \| < \epsilon \]  

(e 0.31)

for all \( f \in \mathcal{F} \). Lemma then follows easily.
Theorem 1.5.
Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection,
Theorem 1.5.

Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection, $C = PM_r(C(X))P$ and let $n \geq 1$ be an integer.
Theorem 1.5.
Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection, $C = PM_r(C(X))P$ and let $n \geq 1$ be an integer. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists $\delta > 0$ and a finite subset $\mathcal{H} \subset C$ satisfying the following:

Suppose that $\phi, \psi : C \to C([0, 1], M_n)$ are two unital homomorphisms such that $\phi^* 0 = \psi^* 0$, $|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta$ for all $g \in \mathcal{H}$, and for all $\tau \in T(C([0, 1], M_n))$. Then there exists a unitary $u \in C([0, 1], M_n)$ such that $\|u^* \phi(f)u - \psi(f)\| < \epsilon$ for all $f \in \mathcal{F}$.
Theorem 1.5.

Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection, $C = PM_r(C(X))P$ and let $n \geq 1$ be an integer. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists $\delta > 0$ and a finite subset $\mathcal{H} \subset C_{s.a.}$ satisfying the following:

Suppose that $\phi, \psi : C \to C([0, 1], M_n)$ are two unital homomorphisms such that $\phi^*0 = \psi^*0$, $|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta$ for all $g \in \mathcal{H}$, and for all $\tau \in \mathcal{T}(C([0, 1], M_n))$.

Then there exists a unitary $u \in C([0, 1], M_n)$ such that $\|u^*\phi(f)u - \psi(f)\| < \epsilon$ for all $f \in \mathcal{F}$. 

(e 0.32)

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Theorem 1.5.
Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection, $C = PM_r(C(X))P$ and let $n \geq 1$ be an integer. For any $\epsilon > 0$ and any finite subset $F \subset C$, there exists $\delta > 0$ and a finite subset $H \subset C_{s.a.}$ satisfying the following: Suppose that $\phi, \psi : C \to C([0,1], M_n)$ are two unital homomorphisms such that
Theorem 1.5.

Let \( X \) be a compact metric space, \( P \in M_r(C(X)) \) be a projection, \( C = PM_r(C(X))P \) and let \( n \geq 1 \) be an integer. For any \( \epsilon > 0 \) and any finite subset \( \mathcal{F} \subset C \), there exists \( \delta > 0 \) and a finite subset \( \mathcal{H} \subset C_{s.a.} \) satisfying the following: Suppose that \( \phi, \psi : C \to C([0, 1], M_n) \) are two unital homomorphisms such that

\[
\phi_{*0} = \psi_{*0}, \quad |\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \quad \text{for all} \quad g \in \mathcal{H}, \quad (e \ 0.32)
\]

and for all \( \tau \in T(C([0, 1], M_n)) \).
**Theorem 1.5.**

Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection, $C = PM_r(C(X))P$ and let $n \geq 1$ be an integer. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists $\delta > 0$ and a finite subset $\mathcal{H} \subset C_{s.a.}$ satisfying the following: Suppose that $\phi, \psi : C \rightarrow C([0,1], M_n)$ are two unital homomorphisms such that

$$\phi_*0 = \psi_*0, \quad |\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{H}, \quad (e \, 0.32)$$

and for all $\tau \in T(C([0,1], M_n))$. Then there exists a unitary $u \in C([0,1], M_n)$ such that

$$\|u^*\phi(f)u - \psi(f)\| < \epsilon \text{ for all } f \in \mathcal{F}. \quad (e \, 0.33)$$
Theorem 1.5.

Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection, $C = PM_r(C(X))P$ and let $n \geq 1$ be an integer. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists $\delta > 0$ and a finite subset $\mathcal{H} \subset C_{s.a.}$ satisfying the following: Suppose that $\phi, \psi : C \to C([0, 1], M_n)$ are two unital homomorphisms such that $\phi_*0 = \psi_*0$, $|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta$ for all $g \in \mathcal{H}$, \hspace{1cm} (e 0.32)

and for all $\tau \in T(C([0, 1], M_n))$. Then there exists a unitary $u \in C([0, 1], M_n)$ such that $\|u^*\phi(f)u - \psi(f)\| < \epsilon$ for all $f \in \mathcal{F}$. \hspace{1cm} (e 0.33)
**Proof**: Let $\delta > 0$ be required by Cor. 1.2. for the given $\epsilon/16$ and $\mathcal{F}$ and $n$. Let $\epsilon_1 = \min\{\epsilon/64, \delta/16\}$. Let $\delta_1 > 0$ (in place of $\delta$) and $H \subset C_{\mathcal{F}} a$. be finite subset as required by Thm. 1.4. for given $\epsilon_1$ (in place of $\epsilon$) and $\mathcal{F}$ (as well as $n$).

Choose $\eta > 0$ such that $\|\phi(f)(t) - \phi(f)(t')\| < \epsilon_1$ and $\|\psi(f)(t) - \psi(f)(t')\| < \epsilon_1$ for all $f \in \mathcal{F}$, whenever $|t - t'| < \eta$.

Let $0 = t_0 < t_1 < \cdots < t_m = 1$ be a partition of $[0, 1]$ with $|t_i - t_{i-1}| < \eta$ for all $i$. By the assumption and 1.4, there is a unitary $u_i \in M_n$ such that $\|u_i^* \phi(f)(t_i) u_i - \psi(f)(t_i)\| < \epsilon_1$ for all $f \in \mathcal{F}$, $i = 0, 1, 2, \ldots, m$.

It follows that $u_i^* \phi(f)(t_i) u_i + u_i^* \psi(f)(t_i) u_i^* \approx \epsilon_1$ for all $f \in \mathcal{F}$, $i = 0, 1, 2, \ldots, m$. (e 0.35)
Proof : Let $\delta > 0$ be required by Cor. 1.2. for the given $\epsilon/16$ and $\mathcal{F}$ and $n$. Let $\epsilon_1 = \min\{\epsilon/64, \delta/16\}$. 
Proof: Let $\delta > 0$ be required by Cor. 1.2. for the given $\epsilon/16$ and $F$ and $n$. Let $\epsilon_1 = \min\{\epsilon/64, \delta/16\}$. Let $\delta_1 > 0$ (in place of $\delta$) and $\mathcal{H} \subset C_{s.a.}$ be finite subset as required by Thm. 1.4. for given $\epsilon_1$ (in place of $\epsilon$) and $F$ (as well as $n$).
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$$\|\phi(f)(t) - \phi(f)(t')\| < \epsilon_1 \text{ and } \|\psi(f)(t) - \psi(f)(t')\| < \epsilon_1$$

(e 0.34)

for all $f \in F$, 

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Proof : Let $\delta > 0$ be required by Cor. 1.2. for the given $\epsilon/16$ and $F$ and $n$. Let $\epsilon_1 = \min\{\epsilon/64, \delta/16\}$. Let $\delta_1 > 0$ (in place of $\delta$) and $\mathcal{H} \subset C_{s.a.}$ be finite subset as required by Thm. 1.4. for given $\epsilon_1$ (in place of $\epsilon$) and $F$ (as well as $n$). Choose $\eta > 0$ such that

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for all $f \in \mathcal{F}$, whenever $|t - t'| < \eta$. 
Proof: Let \( \delta > 0 \) be required by Cor. 1.2. for the given \( \epsilon/16 \) and \( \mathcal{F} \) and \( n \). Let \( \epsilon_1 = \min\{\epsilon/64, \delta/16\} \). Let \( \delta_1 > 0 \) (in place of \( \delta \)) and \( \mathcal{H} \subset C_{s.a.} \) be finite subset as required by Thm. 1.4. for given \( \epsilon_1 \) (in place of \( \epsilon \)) and \( \mathcal{F} \) (as well as \( n \)). Choose \( \eta > 0 \) such that

\[
\|\phi(f)(t) - \phi(f)(t')\| < \epsilon_1 \text{ and } \|\psi(f)(t) - \psi(f)(t')\| < \epsilon_1 \quad (e\,0.34)
\]

for all \( f \in \mathcal{F} \), whenever \( |t - t'| < \eta \). Let \( 0 = t_0 < t_1 < \cdots < t_m = 1 \) be a partition of \([0,1]\) with \( |t_i - t_{i-1}| < \eta \) for all \( i \).
Proof: Let $\delta > 0$ be required by Cor. 1.2. for the given $\epsilon/16$ and $F$ and $n$. Let $\epsilon_1 = \min\{\epsilon/64, \delta/16\}$. Let $\delta_1 > 0$ (in place of $\delta$) and $\mathcal{H} \subset C\text{s.a.}$ be finite subset as required by Thm. 1.4. for given $\epsilon_1$ (in place of $\epsilon$) and $F$ (as well as $n$). Choose $\eta > 0$ such that

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Proof: Let $\delta > 0$ be required by Cor. 1.2. for the given $\epsilon/16$ and $F$ and $n$. Let $\epsilon_1 = \min\{\epsilon/64, \delta/16\}$. Let $\delta_1 > 0$ (in place of $\delta$) and $H \subset C_{s.a.}$ be finite subset as required by Thm. 1.4. for given $\epsilon_1$ (in place of $\epsilon$) and $F$ (as well as $n$). Choose $\eta > 0$ such that

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$$\|u_i^* \phi(f)(t_i) u_i - \psi(f)(t_i)\| < \epsilon_1 \quad \text{for all} \quad f \in F, \quad i = 0, 1, 2, \ldots, m.$$  

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Proof: Let $\delta > 0$ be required by Cor. 1.2. for the given $\epsilon/16$ and $\mathcal{F}$ and $n$. Let $\epsilon_1 = \min \{ \epsilon/64, \delta/16 \}$. Let $\delta_1 > 0$ (in place of $\delta$) and $\mathcal{H} \subset C_{s.a.}$ be finite subset as required by Thm. 1.4. for given $\epsilon_1$ (in place of $\epsilon$) and $\mathcal{F}$ (as well as $n$). Choose $\eta > 0$ such that

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for all $f \in \mathcal{F}$, whenever $|t - t'| < \eta$. Let $0 = t_0 < t_1 < \cdots < t_m = 1$ be a partition of $[0, 1]$ with $|t_i - t_{i-1}| < \eta$ for all $i$. By the assumption and Thm. 1.4, there is a unitary $u_i \in M_n$ such that

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It follows that

$$u_{i+1} u_i^* \phi(f)(t_i) u_i u_{i+1}^* \approx \epsilon_1 \quad u_{i+1} \psi(f)(t_i) u_{i+1}^*$$
**Proof** : Let $\delta > 0$ be required by Cor. 1.2. for the given $\varepsilon/16$ and $\mathcal{F}$ and $n$. Let $\varepsilon_1 = \min\{\varepsilon/64, \delta/16\}$. Let $\delta_1 > 0$ (in place of $\delta$) and $\mathcal{H} \subset C_{s.a.}$ be finite subset as required by Thm. 1.4. for given $\varepsilon_1$ (in place of $\varepsilon$) and $\mathcal{F}$ (as well as $n$). Choose $\eta > 0$ such that

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It follows that

$$u_{i+1}u_i^*\phi(f)(t_i)u_iu_{i+1}^* \approx \varepsilon_1 \quad u_{i+1}\psi(f)(t_i)u_{i+1}^* \approx \varepsilon_1$$

$\approx \varepsilon_1 \quad u_{i+1}\psi(f)(t_{i+1})u_{i+1}^* \approx \varepsilon_1 \quad \phi(f)(t_{i+1}) \approx \varepsilon_1 \quad \phi(f)(t_i).$
Proof: Let $\delta > 0$ be required by Cor. 1.2. for the given $\epsilon/16$ and $\mathcal{F}$ and $n$. Let $\epsilon_1 = \min\{\epsilon/64, \delta/16\}$. Let $\delta_1 > 0$ (in place of $\delta$) and $\mathcal{H} \subset C_{s.a.}$ be finite subset as required by Thm. 1.4. for given $\epsilon_1$ (in place of $\epsilon$) and $\mathcal{F}$ (as well as $n$). Choose $\eta > 0$ such that

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It follows that

$$u_{i+1} u_i^* \phi(f)(t_i) u_i u_{i+1}^* \approx_{\epsilon_1} u_{i+1} \psi(f)(t_i) u_{i+1}^*$$

$$\approx_{\epsilon_1} u_{i+1} \psi(f)(t_{i+1}) u_{i+1}^* \approx_{\epsilon_1} \phi(f)(t_{i+1}) \approx_{\epsilon_1} \phi(f)(t_i).$$
It follows that there exists a continuous path of unitaries 
\[ \{ w_i(t) : t \in [t_i, t_{i+1}] \} \subset M_n \]
It follows that there exists a continuous path of unitaries \( \{ w_i(t) : t \in [t_i, t_{i+1}] \} \subset M_n \) such that \( w_i(t_i) = 1_{M_n} \) and \( w_i(t_{i+1}) = u_{i+1}u_i^* \).
It follows that there exists a continuous path of unitaries
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\[
\|w_i(t)\phi(f)(t_i) - \phi(f)(t_i)w_i(t)\| < \epsilon/16 \quad \text{for all } f \in \mathcal{F}, \quad (e\ 0.36)
\]

\( i = 0, 1, 2..., m. \)
It follows that there exists a continuous path of unitaries 
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\( i = 0, 1, 2..., m \). Define \( v(t) = w_i(t)u_i \) for \( t \in [t_i, t_{i+1}] \), \( i = 0, 1, 2..., m \).
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i = 0, 1, 2..., m. Define \( v(t) = w_i(t) u_i \) for \( t \in [t_i, t_{i+1}] \), \( i = 0, 1, 2..., m \).
Then \( v(t_i) = u_i \) and \( v(t_{i+1}) = u_{i+1}, i = 0, 1, 2, ..., m \), and
\( v \in C([0, 1], M_n) \).
It follows that there exists a continuous path of unitaries 
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\[\|w_i(t)\phi(f)(t_i) - \phi(f)(t_i)w_i(t)\| < \epsilon/16\] for all \(f ∈ F,\) \hspace{1cm} (e 0.36)

\(i = 0, 1, 2..., m.\) Define \(v(t) = w_i(t)u_i\) for \(t ∈ [t_i, t_{i+1}], i = 0, 1, 2..., m.\)
Then \(v(t_i) = u_i\) and \(v(t_{i+1}) = u_{i+1}, i = 0, 1, 2..., m,\) and
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It follows that there exists a continuous path of unitaries 
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\[
\| w_i(t) \phi(f)(t_i) - \phi(f)(t_i) w_i(t) \| < \epsilon/16 \quad \text{for all } f \in \mathcal{F}, \quad (e\ 0.36)
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\( i = 0, 1, 2..., m \). Define \( v(t) = w_i(t) u_i \) for \( t \in [t_i, t_{i+1}], \ i = 0, 1, 2..., m \). Then \( v(t_i) = u_i \) and \( v(t_{i+1}) = u_{i+1}, \ i = 0, 1, 2, ..., m \), and 
\( v \in C([0, 1], M_n) \). Moreover, for \( t \in [t_i, t_{i+1}] \),

\[
v(t)^* \phi(f)(t) v(t) \approx \epsilon_1 \ u_i^* w_i(t)^* \phi(f)(t_i) w_i(t) u_i
\]
It follows that there exists a continuous path of unitaries 
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\( w_i(t_{i+1}) = u_{i+1}u_i^* \) and 
\[ \| w_i(t)\phi(f)(t_i) - \phi(f)(t_i)w_i(t) \| < \epsilon/16 \text{ for all } f \in \mathcal{F}, \] (e 0.36)
i = 0, 1, 2..., m. Define \( v(t) = w_i(t)u_i \) for \( t \in [t_i, t_{i+1}] \), \( i = 0, 1, 2..., m \). Then \( v(t_i) = u_i \) and \( v(t_{i+1}) = u_{i+1} \), \( i = 0, 1, 2,..., m \), and 
\( v \in C([0, 1], M_n) \). Moreover, for \( t \in [t_i, t_{i+1}] \), 
\[ v(t)^*\phi(f)(t)v(t) \approx \epsilon_1 \  u_i^*w_i(t)^*\phi(f)(t_i)w_i(t)u_i \approx \epsilon/16 \  u_i^*\phi(f)(t_i)u_i \]
It follows that there exists a continuous path of unitaries 
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\( w_i(t_i) = 1_{M_n} \) and 
\( w_i(t_{i+1}) = u_{i+1}u_i^* \) and 
\[ \| w_i(t)\phi(f)(t_i) - \phi(f)(t_i)w_i(t) \| < \epsilon/16 \text{ for all } f \in \mathcal{F} \] (e 0.36)

for all \( i = 0, 1, 2..., m \). Define \( v(t) = w_i(t)u_i \) for \( t \in [t_i, t_{i+1}] \), \( i = 0, 1, 2..., m \). Then 
\( v(t_i) = u_i \) and \( v(t_{i+1}) = u_{i+1} \), \( i = 0, 1, 2..., m \), and 
\( v \in C([0,1], M_n) \). Moreover, for \( t \in [t_i, t_{i+1}] \),

\[ v(t)^*\phi(f)(t)v(t) \approx_{\epsilon_1} u_i^* w_i(t)^* \phi(f)(t_i)w_i(t)u_i \approx_{\epsilon/16} u_i^* \phi(f)(t_i)u_i \]
\[ \approx_{\epsilon_1} \psi(t)(t_i) \approx_{\epsilon_1} \psi(f)(t) \]

for all \( f \in \mathcal{F} \).
It follows that there exists a continuous path of unitaries \( \{w_i(t) : t \in [t_i, t_{i+1}]\} \subset M_n \) such that \( w_i(t_i) = \mathbf{1}_{M_n} \) and \( w_i(t_{i+1}) = u_{i+1}u_i^* \) and

\[
\|w_i(t_i)\phi(f)(t_i) - \phi(f)(t_i)w_i(t)\| < \epsilon/16 \quad \text{for all} \quad f \in \mathcal{F}, \quad (e\,0.36)
\]

\( i = 0, 1, 2, \ldots, m \). Define \( v(t) = w_i(t)u_i \) for \( t \in [t_i, t_{i+1}] \), \( i = 0, 1, 2, \ldots, m \). Then \( v(t_i) = u_i \) and \( v(t_{i+1}) = u_{i+1} \), \( i = 0, 1, 2, \ldots, m \), and \( v \in C([0,1], M_n) \). Moreover, for \( t \in [t_i, t_{i+1}] \),

\[
v(t)^*\phi(f)(t)v(t) \approx_{\epsilon_1} u_i^* w_i(t)^*\phi(f)(t_i)w_i(t)u_i \approx_{\epsilon/16} u_i^*\phi(f)(t_i)u_i \\
\approx_{\epsilon_1} \psi(t)(t_i) \approx_{\epsilon_1} \psi(f)(t)
\]

for all \( f \in \mathcal{F} \). In other words,

\[
\|v^*\phi(f)v - \psi(f)\| < \epsilon \quad \text{for all} \quad f \in \mathcal{F}. \quad (e\,0.37)
\]
**Theorem 1.6.**

Let $X$ be a compact metric space which is locally path connected, let $P \in M_r(C(X))$ be a projection and let $C = PM_r(C(X))P$. 

Suppose that $\varphi : C \to C([0,1], M_n)$, where $n \geq 1$ is an integer.

For any $\epsilon > 0$ and any finite subset $F \subset C$, there exists a set of mutually orthogonal rank projections $p_1, p_2, \ldots, p_n \in C([0,1], M_n)$ such that

$$\|\varphi(f) - \sum_{i=1}^{n} f(\alpha_i)p_i\| < \epsilon$$

for all $f \in F$, where $\alpha_i : [0,1] \to X$ is a continuous map, $i = 1, 2, \ldots, n$. 

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Theorem 1.6.

Let $X$ be a compact metric space which is locally path connected, let $P \in M_r(C(X))$ be a projection and let $C = PM_r(C(X))P$. Suppose that $\phi : C \to C([0, 1], M_n)$, where $n \geq 1$ is an integer. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists a set of mutually orthogonal rank projections $p_1, p_2, \ldots, p_n \in C([0, 1], M_n)$ such that

$$\|\phi(f) - \sum_{i=1}^{n} f(\alpha_i) p_i\| < \epsilon$$

for all $f \in \mathcal{F}$, where $\alpha_i : [0, 1] \to X$ is a continuous map, $i = 1, 2, \ldots, n$. 

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$$\|\phi(f) - \sum_{i=1}^{n} f(\alpha_i)p_i\| < \epsilon \text{ for all } f \in \mathcal{F},$$

(e 0.38)
Theorem 1.6.

Let $X$ be a compact metric space which is locally path connected, let $P \in M_r(C(X))$ be a projection and let $C = PM_r(C(X))P$. Suppose that $\phi : C \to C([0,1], M_n)$, where $n \geq 1$ is an integer. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists a set of mutually orthogonal rank projections $p_1, p_2, \ldots, p_n \in C([0,1], M_n)$ such that

$$
\|\phi(f) - \sum_{i=1}^{n} f(\alpha_i)p_i\| < \epsilon \text{ for all } f \in \mathcal{F},
$$

where $\alpha_i : [0, 1] \to X$ is a continuous map, $i = 1, 2, \ldots, n$. 

(e 0.38)
Proof: We will only prove the case that $C = C(X)$. 

Let $\delta > 0$ be required by Lemma 1.3. for the given integer $n$ and $\epsilon/4$ (in place of $\epsilon$).

Let $d > 0$ satisfying the following:

$\|f(x) - f(x')\| < \epsilon/4$ for all $f \in F$, if $\text{dist}(x, x') < 2d$,

and if $\text{dist}(x, y) < d/2$, there exists an open ball $B$ of radius $< d$ which contains a continuous path in $B$ connecting $x$ and $y$.

Let $\delta_1 > 0$ (in place of $\delta$) and $H \subset C$ be a finite subset required by Theorem 1.4 for the given $\min\{\epsilon/4, \delta/2\}$, $F$, $n$ and $d/2$.

There exists $\eta > 0$ such that $\|\phi(g)(t) - \phi(g)(t')\| < \min\{\epsilon/4, \delta_1/2, \delta/2\}$ for all $f \in H$ whenever $|t - t'| < \eta$.

Let $0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$ be a partition with $|t_i - t_{i-1}| < \eta$, $i = 1, 2, \ldots, m$.

We have $\phi(f)(t_i - 1) = \sum_{j=1}^n f(x_{i-1}, j)p_{i-1, j}$ for all $f \in C(X)$, where $x_{i-1}, j \in X$ and $\{p_{i-1, 1}, p_{i-1, 2}, \ldots, p_{i-1, n}\}$ is a set of mutually orthogonal rank one projections.
Proof: We will only prove the case that $C = C(X)$. Let $\delta > 0$ be required by Lemma 1.3 for the given integer $n$ and $\epsilon/4$ (in place of $\epsilon$).
Proof: We will only prove the case that $C = C(X)$. Let $\delta > 0$ be required by Lemma 1.3. for the given integer $n$ and $\epsilon/4$ (in place of $\epsilon$). Let $d > 0$ satisfying the following:

$$\|f(x) - f(x')\| < \epsilon/4 \text{ for all } f \in \mathcal{F}, \text{ if } \text{dist}(x, x') < 2d, \quad (e.0.39)$$
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(e 0.39)

and if $\text{dist}(x, y) < d/2$, there exists an open ball $B$ of radius $< d$ which contains a continuous path in $B$ connecting $x$ and $y$. 

\[ \sum_{j=1}^{n} f(x_{i-1}, j)p_{i-1, j} \]  

(e 0.41)
Proof: We will only prove the case that $C = C(X)$. Let $\delta > 0$ be required by Lemma 1.3. for the given integer $n$ and $\epsilon/4$ (in place of $\epsilon$). Let $d > 0$ satisfying the following:

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and if $\text{dist}(x, y) < d/2$, there exists an open ball $B$ of radius $< d$ which contains a continuous path in $B$ connecting $x$ and $y$. Let $\delta_1 > 0$ (in place of $\delta$) and $\mathcal{H} \subset C$ be a finite subset required by Theorem 1.4 for the given $\min\{\epsilon/4, \delta/2\}$ (in place of $\epsilon$), $F$, $n$ and $d/2$. 

{}
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\]

and if \( \text{dist}(x, y) < d/2 \), there exists an open ball \( B \) of radius \( < d \) which contains a continuous path in \( B \) connecting \( x \) and \( y \). Let \( \delta_1 > 0 \) (in place of \( \delta \)) and \( \mathcal{H} \subset C \) be a finite subset required by Theorem 1.4 for the given \( \min\{\epsilon/4, \delta/2\} \) (in place of \( \epsilon \), \( \mathcal{F}, n \) and \( d/2 \). There exists \( \eta > 0 \) such that

\[
| \phi(g)(t) - \phi(g)(t') | < \min\{\epsilon/4, \delta_1/2, \delta/2\} \text{ for all } f \in \mathcal{H} \quad (e\ 0.40)
\]

whenever \( |t - t'| < \eta \).
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$$\phi(f)(t_{i-1}) = \sum_{j=1}^{n} f(x_{i-1,j})p_{i-1,j} \text{ for all } f \in C(X), \quad (e \ 0.41)$$

where $x_{i-1,j} \in X$ and $\{p_{i-1,1}, p_{i-1,2}, ..., p_{i-1,n}\}$ is a set of mutually orthogonal rank one projections.
It follows from Thm. 1.4 and (e 0.40) that there are unitaries $u_i \in M_n$ such that

$$\|u_i^* \phi(f(t_{i-1})) u_i - \phi(f(t_{i-1}))\| < \min\{\delta/2, \epsilon/4\}$$

for all $f \in F$, (e 0.42)

Moreover, we may assume, without loss of generality, that there is a permutation $\sigma_i$ such that

$$u_i^* p_i^{-1}, j u_i = p_i, \sigma_i(j)$$

and

$$\text{dist}(x_i^{-1}, j, x_i, \sigma_i(j)) < d/2,$$

(e 0.43)

for all $j = 1, 2, \ldots, n$, $i = 1, 2, \ldots, m$.

By (e 0.42) and (e 0.40),

$$\|\phi(f(t_{i-1})) u_i - \phi(f(t_{i-1})) u_i^* \phi(f(t_{i-1}))\| < \delta$$

for all $f \in F$, (e 0.44)

It follows from 1.1 that there exists a continuous path of unitaries $\{v(t_i) : t_i \in [t_i-1, t_i]\} \subset M_n$ such that

$$v(t_i) = 1$$

and

$$\|v(t_i) \phi(f(t_{i-1})) - \phi(f(t_{i-1})) v(t_i)\| < \epsilon/4$$

for all $f \in F$, (e 0.45)

$i = 1, 2, \ldots, m$. 
It follows from Thm. 1.4 and (e 0.40) that there are unitaries $u_i \in M_n$ such that

$$\|u_i^* \phi(f)(t_{i-1}) u_i - \phi(f)(t_i)\| < \min\{\delta/2, \epsilon/4\} \text{ for all } f \in \mathcal{F}, \ (e\ 0.42)$$

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\( i = 1, 2, \ldots, m. \) Moreover, we may assume, without loss of generality, that there is a permutation \( \sigma_i \) such that

\[
u_i^* p_{i-1,j} u = p_{i,\sigma_i(j)} \text{ and } \text{dist}(x_{i-1,j}, x_{i,\sigma_i(j)}) < d/2, \quad (e 0.45)\]

\( j = 1, 2, \ldots, n, \quad i = 1, 2, \ldots, m. \)
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$i = 1, 2, \ldots, m$. It follows from 1.1 that there exists a continuous path of unitaries $\{v(t) : t \in [t_{i-1}, t_i]\} \subset M_n$.
It follows from Thm. 1.4 and (e 0.40) that there are unitaries \( u_i \in M_n \) such that
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j = 1, 2, ..., n, i = 1, 2, ..., m. By (e 0.42) and (e 0.40),
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i = 1, 2, ..., m. It follows from 1.1 that there exists a continuous path of unitaries \( \{v(t) : t \in [t_{i-1}, t_i]\} \subset M_n \) such that \( v(t_{i-1}) = 1 \) and \( v(t_i) = u_{i-1} \) and
It follows from Thm. 1.4 and (e 0.40) that there are unitaries \( u_i \in M_n \) such that
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\( i = 1, 2, \ldots, m \). Moreover, we may assume, without loss of generality, that there is a permutation \( \sigma_i \) such that
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\( j = 1, 2, \ldots, n, \quad i = 1, 2, \ldots, m \). By (e 0.42) and (e 0.40),
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\]
\( i = 1, 2, \ldots, m \). It follows from 1.1 that there exists a continuous path of unitaries \( \{ \nu(t) : t \in [t_{i-1}, t_i] \} \subset M_n \) such that \( \nu(t_{i-1}) = 1 \) and \( \nu(t_i) = u_{i-1} \) and
\[
\| \nu(t) \phi(f)(t_{i-1}) - \phi(f)(t_{i-1}) \nu(t) \| < \epsilon/4 \quad \text{for all } f \in \mathcal{F}, \quad (e 0.45)
\]
\( i = 1, 2, \ldots, m \).
Define $p_j(t) = v(t) \ast p_{i-1,j} v(t)$ for $t \in [t_{i-1}, t_i]$, $i = 1, 2, ..., m$. 
Define $p_j(t) = v(t)^* \gamma_{i-1,j} v(t)$ for $t \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, m$. Then $p_j(t_0) = p_{0,j}$, $p_j(t_i) = p_{i,\sigma_{i}(j)}$, $i = 1, 2, \ldots, m$. 
Define $p_j(t) = v(t)*p_{i-1,j}v(t)$ for $t \in [t_{i-1}, t_i]$, $i = 1, 2, ..., m$. Then

$p_j(t_0) = p_{0,j}$, $p_j(t_i) = p_{i,\sigma_i(j)}$, $i = 1, 2, ..., m$. Since

$\text{dist}(x_{i-1,j}, x_{i,\sigma_i(j)}) < d/2$, 


Define $p_j(t) = v(t)^* p_{i-1,j} v(t)$ for $t \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, m$. Then

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$\alpha_{j,i-1} : [t_{i-1}, t_i] \to B_i$ such that
Define \( p_j(t) = \nu(t) \cdot p_{i-1,j} \nu(t) \) for \( t \in [t_{i-1}, t_i] \), \( i = 1, 2, ..., m \). Then \( p_j(t_0) = p_{0,j} \), \( p_j(t_i) = p_{i,\sigma_i(j)} \), \( i = 1, 2, ..., m \). Since \( \text{dist}(x_{i-1,j}, x_{i,\sigma_i(j)}) < d/2 \), there exists a continuous path \( \alpha_{j,i-1} : [t_{i-1}, t_i] \rightarrow B_i \) such that \( \alpha_{j,i-1}(t_{i-1}) = x_{i-1,j} \) and \( \alpha_{j,i-1}(t_i) = x_{i,\sigma_i(j)} \).
Define \( p_j(t) = v(t) \cdot p_{i-1,j} v(t) \) for \( t \in [t_{i-1}, t_i] \), \( i = 1, 2, \ldots, m \). Then 
\( p_j(t_0) = p_{0,j}, p_j(t_i) = p_{i,\sigma_i(j)}, \ i = 1, 2, \ldots, m \). Since 
\( \text{dist}(x_{i-1,j}, x_{i,\sigma_i(j)}) < d/2 \), there exists a continuous path 
\( \alpha_{j,i-1} : [t_{i-1}, t_i] \to B_i \) such that 
\( \alpha_{j,i-1}(t_{i-1}) = x_{i-1,j} \) and 
\( \alpha_{j,i-1}(t_i) = x_{i,\sigma_i(j)} \), where \( B_i \) is an open ball with radius \( d \) which contains both \( x_{i-1,j} \) and \( x_{i,\sigma_i(j)} \).
Define $p_j(t) = v(t) p_{i-1,j} v(t)$ for $t \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, m$. Then $p_j(t_0) = p_{0,j}$, $p_j(t_i) = p_{i,\sigma_i(j)}$, $i = 1, 2, \ldots, m$. Since $\text{dist}(x_{i-1,j}, x_{i,\sigma_i(j)}) < d/2$, there exists a continuous path $\alpha_{j,i-1} : [t_{i-1}, t_i] \to B_i$ such that $\alpha_{j,i-1}(t_{i-1}) = x_{i-1,j}$ and $\alpha_{j,i-1}(t_i) = x_{i,\sigma_i(j)}$, where $B_i$ is an open ball with radius $d$ which contains both $x_{i-1,j}$ and $x_{i,\sigma_i(j)}$. Define $\alpha_j : [0, 1] \to X$ by $\alpha_j(t) = \alpha_{j,i-1}(t)$ if $t \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, m$. 

\[\psi(f) = \sum_{i=1}^{n} f(\alpha_i) p_i \quad \text{for all } f \in C(X)\]
Define \( p_j(t) = \nu(t)^* p_{i-1,j} \nu(t) \) for \( t \in [t_{i-1}, t_i] \), \( i = 1, 2, ..., m \). Then
\[
p_j(t_0) = p_{0,j}, \quad p_j(t_i) = p_{i,\sigma_i(j)}, \quad i = 1, 2, ..., m.
\]
Since
\[
\text{dist}(x_{i-1,j}, x_{i,\sigma_i(j)}) < d/2,
\]
there exists a continuous path
\[
\alpha_{j,i-1} : [t_{i-1}, t_i] \to B_i
\]
such that \( \alpha_{j,i-1}(t_{i-1}) = x_{i-1,j} \) and
\[
\alpha_{j,i-1}(t_i) = x_{i,\sigma_i(j)},
\]
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\]
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\[
\psi(f) = \sum_{i=1}^{n} f(\alpha_i)p_i \quad \text{for all} \quad f \in C(X). \tag{e.0.46}
\]
On $t \in [t_{i-1}, t_i]$,
On $t \in [t_{i-1}, t_i]$, 

$$
\| \phi(f)(t) - \psi(f)(t) \| = \| \phi(f)(t) - \sum_{j=1}^{n} f(x_{i-1,j})p_{i-1,j} \| \\
+ \| \sum_{j=1}^{n} f(x_{i-1,j})p_{i-1,j} - \sum_{j=1}^{n} f(\alpha_{j,i-1}(t))p_{j}(t) \|
$$
On $t \in [t_{i-1}, t_i]$, 

$$\|\phi(f)(t) - \psi(f)(t)\| = \|\phi(f)(t) - \sum_{j=1}^{n} f(x_{i-1,j})p_{i-1,j}\|$$

$$+ \| \sum_{j=1}^{n} f(x_{i-1,j})p_{i-1,j} - \sum_{j=1}^{n} f(\alpha_{j,i-1}(t))p_{j}(t)\|$$

$$< \epsilon/4 + \| \sum_{j=1}^{n} f(x_{i-1,j})p_{i-1,j} - \sum_{j=1}^{n} f(x_{i-1,j})v^*(t)p_{i-1,j}v(t)\| + \epsilon/4$$
On $t \in [t_{i-1}, t_i]$, 

$$\| \phi(f)(t) - \psi(f)(t) \| = \| \phi(f)(t) - \sum_{j=1}^{n} f(x_{i-1,j})p_{i-1,j} \|$$

$$+ \| \sum_{j=1}^{n} f(x_{i-1,j})p_{i-1,j} - \sum_{j=1}^{n} f(\alpha_{j,i-1}(t))p_j(t) \|$$

$$< \epsilon/4 + \| \sum_{j=1}^{n} f(x_{i-1,j})p_{i-1,j} - \sum_{j=1}^{n} f(x_{i-1,j})v^*(t)p_{i-1,j}v(t) \| + \epsilon/4$$

$$= \| \phi(f)(t_{i-1}) - v^*(t)\phi(f)(t_{i-1})v(t) \| + \epsilon/2 < \epsilon/2 + \epsilon/2$$

for all $f \in \mathcal{F}$. 
Corollary

Let $X$ be a compact metric space, let $P \in M_r(C(X))$ be a projection and let $C = PM_r(C(X))P$. 
Corollary

Let $X$ be a compact metric space, let $P \in M_r(C(X))$ be a projection and let $C = PM_r(C(X))P$. Suppose that $\phi : C \to C([0, 1], M_n)$, where $n \geq 1$ is an integer.

Proof.
Corollary

Let $X$ be a compact metric space, let $P \in \mathcal{M}_r(C(X))$ be a projection and let $C = P\mathcal{M}_r(C(X))P$. Suppose that $\phi : C \to C([0, 1], M_n)$, where $n \geq 1$ is an integer. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists a set of mutually orthogonal rank projections $p_1, p_2, \ldots, p_n \in C([0, 1], M_n)$ such that

$$\|\phi(f) - \sum_{i=1}^{n} f(\alpha_i) p_i\| < \epsilon$$

for all $f \in \mathcal{F}$, where $\alpha_i : [0, 1] \to X$ is a continuous map, $i = 1, 2, \ldots, n$. 

Proof.
Corollary

Let $X$ be a compact metric space, let $P \in M_r(C(X))$ be a projection and let $C = PM_r(C(X))P$. Suppose that $\phi : C \to C([0, 1], M_n)$, where $n \geq 1$ is an integer. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists a set of mutually orthogonal rank projections $p_1, p_2, \ldots, p_n \in C([0, 1], M_n)$ such that

$$\|\phi(f) - \sum_{i=1}^{n} f(\alpha_i)p_i\| < \epsilon$$

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Corollary

Let $X$ be a compact metric space, let $P \in M_r(C(X))$ be a projection and let $C = PM_r(C(X))P$. Suppose that $\phi : C \to C([0, 1], M_n)$, where $n \geq 1$ is an integer. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists a set of mutually orthogonal rank projections $p_1, p_2, \ldots, p_n \in C([0, 1], M_n)$ such that

$$\|\phi(f) - \sum_{i=1}^{n} f(\alpha_i)p_i\| < \epsilon \text{ for all } f \in \mathcal{F},$$

where $\alpha_i : [0, 1] \to X$ is a continuous map, $i = 1, 2, \ldots, n$. 

Proof.

$C(X) = \lim_{n \to \infty} (C(X_n), \mathfrak{I}_n)$, where $X_n$ is a polygon and $\mathfrak{I}_n$ is an injective homomorphism.
Corollary

Let $X$ be a compact metric space, let $P \in M_r(C(X))$ be a projection and let $C = PM_r(C(X))P$. Suppose that $\phi : C \to C([0,1], M_n)$, where $n \geq 1$ is an integer. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists a set of mutually orthogonal rank projections $p_1, p_2, ..., p_n \in C([0,1], M_n)$ such that

$$\|\phi(f) - \sum_{i=1}^{n} f(\alpha_i)p_i\| < \epsilon \quad \text{for all } f \in \mathcal{F},$$

where $\alpha_i : [0,1] \to X$ is a continuous map, $i = 1, 2, ..., n$.

Proof.

$C(X) = \lim_{n \to \infty} (C(X_n), \iota_n)$, where $X_n$ is a polygon and $\iota_n$ is an injective homomorphism.
Suppose that $u, v \in M_n$ are two unitaries such that $\|uv - vu\| < 1$. 

One has 

$$\left(\frac{1}{2\pi i}\right) \text{Tr}(\log(v^*uvu^*)) \in \mathbb{Z}.$$ 

If there is a continuous path of unitaries $\{v(t) : t \in [0,1]\} \subset M_n$ such that $v(0) = v$ and $v(1) = 1_{M_n}$ and $\|v^*(t)uv(t)u^* - 1\| < 1$, then 

$$\left(\frac{1}{2\pi i}\right) \text{Tr}(\log(v^*(t)uv(t)u^*))$$ 

is continuous and is zero at $t = 1$. Therefore, 

$$\left(\frac{1}{2\pi i}\right) \text{Tr}(\log(v^*(t)uv(t)u^*)) = 0$$ 

for all $t \in [0,1]$. 

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$$(1/2\pi i) Tr(\log(v^*(t)uv(t)u^*)) = 0 \text{ for all } t \in [0, 1].$$
Let

\[ u_n = \begin{pmatrix}
    e^{2\pi i/n} & 0 & 0 & \cdots \\
    0 & e^{4\pi i/n} & 0 & \cdots \\
    & & & \ddots \\
    & & & & e^{2n\pi i/n}
\end{pmatrix} \]
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\[ u_n = \begin{pmatrix}
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    & & & \ddots \\
    & & & e^{2n\pi i/n}
\end{pmatrix} \]

\[ v_n = \begin{pmatrix}
    0 & 0 & 0 & \cdots & 1 \\
    1 & 0 & 0 & \cdots & 0 \\
    0 & 1 & 0 & \cdots & 0 \\
    & & & \ddots & \ddots \\
    & & & & 1 & 0
\end{pmatrix} \]
One computes that

\[ v_n^* u_n v_n u_n^* = \begin{pmatrix} e^{2\pi i/n} & 0 & 0 & \cdots \\ 0 & e^{2\pi i/n} & 0 & \cdots \\ & & \ddots & \vdots \\ & & & e^{2\pi i/n} \end{pmatrix} \]
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\end{pmatrix}$$

In particular

$$\lim_{n \to \infty} \|u_n v_n - v_n u_n\| = \lim_{n \to \infty} |e^{2\pi i/n} - 1| = 0.$$
One computes that

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    \end{pmatrix}
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However,

\[
    Tr(\log(v_n^* u_n v_n u_n^*)) = 2\pi i.
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In other words, there is \( \delta > 0 \) satisfying the following:
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For any integer \( n \geq 1 \), any pair of unitaries \( u, v \in M_n \) with \( \|uv - vu\| < \delta \), there is a continuous path of unitaries \( \{v(t) : t \in [0, 1]\} \subset M_n \) such that \( v(0) = v \) and \( v(1) = 1_{M_n} \).
One computes that

$$v_n^*u_nv_nu_n^* = \begin{pmatrix} e^{2\pi i/n} & 0 & 0 \ldots \\ 0 & e^{2\pi i/n} & 0 \ldots \\ & & & \ddots \\ & & & & e^{2\pi i/n} \end{pmatrix}.$$

In particular

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In other words, there is \(\delta > 0\) satisfying the following:

For any integer \(n \geq 1\), any pair of unitaries \(u, v \in M_n\) with \(\|uv - vu\| < \delta\), there is a continuous path of unitaries \(\{v(t) : t \in [0, 1]\} \subset M_n\) such that \(v(0) = v\) and \(v(1) = 1_{M_n}\) and

$$\|uv(t) - v(t)u\| < 1 \text{ for all } t \in [0, 1].$$
Let

\[ f(e^{2\pi it}) = \begin{cases} 
1 - 2t, & \text{if } 0 \leq t \leq 1/2; \\
-1 + 2t, & \text{if } 1/2 < t \leq 1,
\end{cases} \]
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\[ g(e^{2\pi it}) = \begin{cases} 
(f(e^{2\pi it} - f(e^{2\pi it})^2)^{1/2}, & \text{if } 0 \leq t \leq 1/2; \\
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These are non-negative continuous functions defined on \( \mathbb{T} \).
Let
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These are non-negative continuous functions defined on \( \mathbb{T} \). Suppose that \( u \) and \( v \) are unitaries with \( uv = vu \). Define

\[ e(u, v) = \begin{pmatrix} f(v) & g(v) + h(v)u^* \\
g(v) + uh(v) & 1 - f(v) \end{pmatrix}. \]
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Then \( e(u, v) \) is a projection.
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There exists a $\delta_0 > 0$ such that if $\|uv - vu\| < \delta_0$, 

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There exists a $\delta_0 > 0$ such that if $\|uv - vu\| < \delta_0$, then the spectrum of positive element $e(u, v)$ has a gap at $1/2$. The bott element $\text{bott}_1(u, v)$ as defined by Exel and Loring is

$$[\chi_{[1/2, \infty]}(e(u, v))] - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$
Let \( C \) be a unital \( C^* \)-algebra \( C \).
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Let $C$ be a unital $C^*$-algebra $C$. Denote by $T(C)$ the tracial state space of $C$. Denote by $\text{Aff}(T(C))$ the space of all real continuous affine functions on $T(C)$. Suppose that $T(C) \neq \emptyset$. There is a map $c \mapsto \hat{c}$ from $C_{s.a.} \to \text{Aff}(C)$ defined by $\hat{c}(\tau) = \tau(c)$ for all $c \in C_{s.a.}$ and $\tau \in T(C)$. 

Denote by $C^q_+$ the image of $C_+^+$ in $\text{Aff}(T(C))$ and $C^q_++$ the image of $C_+^+$ in the unit ball of $C$. Let $A$ and $B$ be two unital $C^*$-algebras and let $L : A \to B$ be a linear map. Let $G \subset A$ be a subset and let $\delta > 0$. We say $L$ is $G$-$\delta$-multiplicative, if $\|L(a)L(b) - L(ab)\| < \delta$ for all $a, b \in G$. 

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Let $C$ be a unital $C^*$-algebra $C$. Denote by $T(C)$ the tracial state space of $C$. Denote by $\text{Aff}(T(C))$ the space of all real continuous affine functions on $T(C)$. Suppose that $T(C) \neq \emptyset$. There is a map $c \mapsto \hat{c}$ from $C_{s.a.} \rightarrow \text{Aff}(C)$ defined by $\hat{c}(\tau) = \tau(c)$ for all $c \in C_{s.a.}$ and $\tau \in T(C)$. Denote by $C_+^q$ the image of $C_+$ in $\text{Aff}(T(C))$ and $C_+^{q,1}$ the image of $C_+$ in the unit ball of $C$. 
Let $C$ be a unital $C^*$-algebra $C$. Denote by $T(C)$ the tracial state space of $C$. Denote by $\text{Aff}(T(C))$ the space of all real continuous affine functions on $T(C)$. Suppose that $T(C) \neq \emptyset$. There is a map $c \mapsto \hat{c}$ from $C_{s.a.} \rightarrow \text{Aff}(C)$ defined by $\hat{c}(\tau) = \tau(c)$ for all $c \in C_{s.a.}$ and $\tau \in T(C)$. Denote by $C^q_+$ the image of $C_+$ in $\text{Aff}(T(C))$ and $C^q_{+,1}$ the image of $C_+$ in the unit ball of $C$.

Let $A$ and $B$ be two unital $C^*$-algebras and let $L : A \rightarrow B$ be a linear map.
Let $C$ be a unital $C^*$-algebra $C$. Denote by $T(C)$ the tracial state space of $C$. Denote by $\text{Aff}(T(C))$ the space of all real continuous affine functions on $T(C)$. Suppose that $T(C) \neq \emptyset$. There is a map $c \mapsto \hat{c}$ from $C_{s.a.} \to \text{Aff}(C)$ defined by $\hat{c}(\tau) = \tau(c)$ for all $c \in C_{s.a.}$ and $\tau \in T(C)$. Denote by $C^q_+$ the image of $C_+$ in $\text{Aff}(T(C))$ and $C^q_{+,1}$ the image of $C_+$ in the unit ball of $C$.

Let $A$ and $B$ be two unital $C^*$-algebras and let $L : A \to B$ be a linear map. Let $G \subset A$ be a subset and let $\delta > 0$. 
Let $C$ be a unital $C^*$-algebra $C$. Denote by $T(C)$ the tracial state space of $C$. Denote by $\text{Aff}(T(C))$ the space of all real continuous affine functions on $T(C)$. Suppose that $T(C) \neq \emptyset$. There is a map $c \mapsto \hat{c}$ from $C_{s.a.} \rightarrow \text{Aff}(C)$ defined by $\hat{c}(\tau) = \tau(c)$ for all $c \in C_{s.a.}$ and $\tau \in T(C)$. Denote by $C^q_+$ the image of $C_+$ in $\text{Aff}(T(C))$ and $C^q_{+1}$ the image of $C_+$ in the unit ball of $C$.

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Let $C$ be a unital $C^*$-algebra $C$. Denote by $T(C)$ the tracial state space of $C$. Denote by $\text{Aff}(T(C))$ the space of all real continuous affine functions on $T(C)$. Suppose that $T(C) \neq \emptyset$. There is a map $c \mapsto \hat{c}$ from $C_{s.a.} \to \text{Aff}(C)$ defined by $\hat{c}(\tau) = \tau(c)$ for all $c \in C_{s.a.}$ and $\tau \in T(C)$.

Denote by $C^+_q$ the image of $C_+$ in $\text{Aff}(T(C))$ and $C^+_q,1$ the image of $C_+$ in the unit ball of $C$.

Let $A$ and $B$ be two unital $C^*$-algebras and let $L : A \to B$ be a linear map. Let $G \subset A$ be a subset and let $\delta > 0$. We say $L$ is $G$-$\delta$-multiplicative, if

$$\|L(a)L(b) - L(ab)\| < \delta \text{ for all } a, b \in G.$$
Now we will present the following theorem:

Let $X$ be a compact metric space, $P \in \mathcal{M}_r(C(X))$ be a projection and $C = P \mathcal{M}_r(C(X))$. Let $\Delta : C_{q,1} + \{0\} \to (0,1)$ be an order preserving map. Let $\epsilon > 0$ and let $F \subset A$ be a finite subset. There exists a finite subset $H_1 \subset A + \{0\}$, a finite subset $G \subset A$, $\delta > 0$, a finite subset $P \subset K(A)$, a finite subset $H_2 \subset A$ and $\sigma > 0$ satisfying the following: Suppose that $L_1, L_2 : A \to \mathcal{M}_k$ (for some integer $k \geq 1$) are two unital $G$-\(\delta\)-multiplicative contractive completely positive linear maps such that

$$\|L_1|_P - L_2|_P\| \leq \sigma$$

for all $h \in H_2$, then there exists a unitary $u \in \mathcal{M}_k$ such that

$$\|\text{Ad}_u \circ L_1(f) - L_2(f)\| < \epsilon$$

for all $f \in F$. (e 10.48)
Now we will present the following theorem:

**Theorem 2.1.**

Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection
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Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. Let $\Delta : C^q_{+} \setminus \{0\} \to (0, 1)$ be an order preserving map.
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Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. Let $\Delta : C^q_{+1} \setminus \{0\} \to (0, 1)$ be an order preserving map. Let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. There exists a finite subset $\mathcal{H}_1 \subset A_+ \setminus \{0\}$,
Theorem 2.1.

Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. Let $\Delta : C^q_+ \setminus \{0\} \to (0,1)$ be an order preserving map. Let $\epsilon > 0$ and let $F \subset A$ be a finite subset. There exists a finite subset $H_1 \subset A_+ \setminus \{0\}$, a finite subset $G \subset A$, a finite subset $P \subset K(A)$, a finite subset $H_2 \subset A$ and $\sigma > 0$ satisfying the following:

Suppose that $L_1, L_2 : A \to M_k$ (for some integer $k \geq 1$) are two unital $G$- multiplicative contractive completely positive linear maps such that $\|L_1|_P = \|L_2|_P$, $\text{tr} \circ L_1(h) \geq \Delta(\hat{h})$, $\text{tr} \circ L_2(h) \geq \Delta(\hat{h})$ for all $h \in H_1$ and $\|\text{tr} \circ L_1(h) - \text{tr} \circ L_2(h)\| < \sigma$ for all $h \in H_2$, then there exists a unitary $u \in M_k$ such that $\|\text{Ad}_u \circ L_1(f) - L_2(f)\| < \epsilon$ for all $f \in F$. (e 10.48)
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Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. Let $\Delta : C^q_+ \setminus \{0\} \to (0, 1)$ be an order preserving map. Let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. There exists a finite subset $\mathcal{H}_1 \subset A_+ \setminus \{0\}$, a finite subset $\mathcal{G} \subset A$, $\delta > 0$, a finite subset $\mathcal{P} \subset K(A)$,
Now we will present the following theorem:

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Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. Let $\Delta : C^q_{+} \setminus \{0\} \to (0, 1)$ be an order preserving map. Let $\epsilon > 0$ and let $F \subset A$ be a finite subset. There exists a finite subset $H_1 \subset A_+ \setminus \{0\}$, a finite subset $G \subset A$, $\delta > 0$, a finite subset $P \subset K(A)$, a finite subset $H_2 \subset A_{s.a.}$ and $\sigma > 0$ satisfying the following:
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$$\text{tr} \circ L_1(h) \geq \Delta(\hat{h}), \quad \text{tr} \circ L_2(h) \geq \Delta(\hat{h}) \quad \text{for all} \ h \in \mathcal{H}_1$$
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and

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Now we will present the following theorem:

**Theorem 2.1.**
Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. Let $\Delta : C^{q,1}_+ \setminus \{0\} \rightarrow (0,1)$ be an order preserving map. Let $\epsilon > 0$ and let $F \subset A$ be a finite subset. There exists a finite subset $H_1 \subset A_+ \setminus \{0\}$, a finite subset $\mathcal{G} \subset A$, $\delta > 0$, a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ and $\sigma > 0$ satisfying the following: Suppose that $L_1, L_2 : A \rightarrow M_k$ (for some integer $k \geq 1$) are two unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps such that

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\]

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\]

then there exists a unitary $u \in M_k$ such that

\[
\|Ad_u \circ L_1(f) - L_2(f)\| < \epsilon \quad \text{for all } f \in F.
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Now we will present the following theorem:

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Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. Let $\Delta : C^q_{+} \setminus \{0\} \to (0,1)$ be an order preserving map. Let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. There exists a finite subset $\mathcal{H}_1 \subset A_+ \setminus \{0\}$, a finite subset $\mathcal{G} \subset A$, $\delta > 0$, a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ and $\sigma > 0$ satisfying the following: Suppose that $L_1, L_2 : A \to M_k$ (for some integer $k \geq 1$) are two unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps such that

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\]
We begin with the following:

**Theorem 2.2.**

Let $X$ be a connected compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))P$. 

Proof. The proof is just a modification of that of Theorem 1.4.
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Let $X$ be a connected compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))P$. Let $\mathcal{F} \subseteq C$ be a finite subset, and let $\epsilon > 0$ be a constant.

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$$\|\phi(f) - u^* \psi(f) u\| < \epsilon \quad \text{for any } f \in \mathcal{F}.$$
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Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = P M_r(C(X))P$ and let $\Delta : A_{+}^{q,1} \backslash \{0\} \to (0,1)$ be an order preserving map. For any $\epsilon > 0$, any finite subset $F \subset A$, there exists a finite subset $P$ of projections in $C$,
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then, there exist a unitary $u \in M_n$ such that

$$\|Ad_u \circ \phi_1(f) - \phi_2(f)\| < \epsilon \text{ for all } f \in \mathcal{F}.$$
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$$|\tau \circ \phi_1(g) - \tau \circ \phi_2(g)| < \sigma \text{ for all } g \in \mathcal{H}_2,$$

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$$\|Ad u \circ \phi_1(f) - \phi_2(f)\| < \epsilon \text{ for all } f \in \mathcal{F}. \quad (e\, 10.49)$$
Proof.
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In the case that \( X \) is connected, then it follows immediately from the previous theorem. Then it is clear that the case \( X \) has finitely many connected components follows. The general case follows from the fact that \( C(X) = \lim_{n \to \infty} (C(X_n), \iota_n) \), where \( X_n \) is a polygon and \( \iota_n \) is injective.

Remark: \( \mathcal{P} \) can be chosen to be a set of mutually orthogonal projections which corresponds to a set of disjoint clopen subsets with union \( X \).
Lemma 2.4.

Let $X$ be a compact metric space
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Lemma 2.4. Let $X$ be a compact metric space and let $A = PM_r(C(X))P$, where $P \in C(X, M_n)$ is a projection. Let $\Delta : A^{q,1}_1 \setminus \{0\} \rightarrow (0,1)$ be an order preserving map.
Lemma 2.4.

Let $X$ be a compact metric space and let $A = PM_r(C(X))P$, where $P \in C(X, M_n)$ is a projection. Let $\Delta : A_1^{q,1} \setminus \{0\} \to (0,1)$ be an order preserving map. For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $\sigma > 0$, there exists a finite subset $H_1 \subset A_1^{q,1} \setminus \{0\}$, a finite subset $H_2 \subset A_1^{q,1} \setminus \{0\}$ and a unitary $u \in M_n$ such that

$$
\| Ad_u \circ \phi_1(f) - (h_1(f) + H(f)) \| < \epsilon,
$$

$$
\| \phi_2(f) - (h_2(f) + H(f)) \| < \epsilon
$$

for all $f \in \mathcal{F}$ and $\tau(1 - p) < \sigma$, where $\tau$ is the tracial state of $M_n$. 

Huaxin Lin

Basic Homotopy Lemmas Introduction

June 8th, 2015, RMMC/CBMS University of Wyoming
Lemma 2.4.

Let $X$ be a compact metric space and let $A = PM_r(C(X))P$, where $P \in C(X, M_n)$ is a projection. Let $\Delta : A^q_1 \setminus \{0\} \to (0, 1)$ be an order preserving map. For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $\sigma > 0$, there exists a finite subset $\mathcal{H}_1 \subset A^1_+ \setminus \{0\}$,
Lemma 2.4.

Let $X$ be a compact metric space and let $A = PM_r(C(X))P$, where $P \in C(X, M_n)$ is a projection. Let $\Delta : A_{+1}^q \setminus \{0\} \to (0, 1)$ be an order preserving map. For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $\sigma > 0$, there exists a finite subset $\mathcal{H}_1 \subset A_{+,1}^1 \setminus \{0\}$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ and $\delta > 0$ satisfying the following:
Lemma 2.4.

Let $X$ be a compact metric space and let $A = PM_r(C(X))P$, where $P \in C(X, M_n)$ is a projection. Let $\Delta : A_{1}^{\mathbb{Q}, 1} \setminus \{0\} \to (0, 1)$ be an order preserving map. For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $\sigma > 0$, there exists a finite subset $\mathcal{H}_1 \subset A_{+}^{1} \setminus \{0\}$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ and $\delta > 0$ satisfying the following: If $\phi_1, \phi_2 : A \to M_n$ (for some integer $n \geq 1$) are two unital homomorphisms
Lemma 2.4.

Let $X$ be a compact metric space and let $A = PM_r(C(X))P$, where $P \in C(X, M_n)$ is a projection. Let $\Delta : A_1^{q,1} \setminus \{0\} \to (0,1)$ be an order preserving map. For any $\epsilon > 0$, any finite subset $F \subset A$ and any $\sigma > 0$, there exists a finite subset $\mathcal{H}_1 \subset A_+ \setminus \{0\}$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ and $\delta > 0$ satisfying the following: If $\phi_1, \phi_2 : A \to M_n$ (for some integer $n \geq 1$) are two unital homomorphisms such that

$$\tau \circ \phi_1(h) \geq \Delta(\hat{h}) \quad \text{for all } h \in \mathcal{H}_1$$

and
Lemma 2.4.

Let $X$ be a compact metric space and let $A = PM_r(C(X))P$, where $P \in C(X, M_n)$ is a projection. Let $\Delta : A^{q,1}_+ \setminus \{0\} \to (0,1)$ be an order preserving map. For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $\sigma > 0$, there exists a finite subset $\mathcal{H}_1 \subset A^{1}_+ \setminus \{0\}$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ and $\delta > 0$ satisfying the following: If $\phi_1, \phi_2 : A \to M_n$ (for some integer $n \geq 1$) are two unital homomorphisms such that

$$\tau \circ \phi_1(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1 \text{ and }$$

$$|\tau \circ \phi_1(g) - \tau \circ \phi_2(g)| < \sigma \text{ for all } g \in \mathcal{H}_2,$$

then, there exist a projection $p \in M_n$, a unital homomorphism $h : A \to pM_n p$, unital homomorphisms $h_1, h_2 : A \to (1 - p)M_n(1 - p)$ and a unitary $u \in M_n$ such that

$$\|Ad u \circ \phi_1(f) - (h_1(f) + h(f))\| < \epsilon,$$

$$\|\phi_2(f) - (h_2(f) + h(f))\| < \epsilon$$

for all $f \in \mathcal{F}$ and $\tau(1 - p) < \sigma$. 

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Huaxin Lin

Basic Homotopy Lemmas Introduction

June 8th, 2015, RMMC/CBMS University of Wyoming
Lemma 2.4.

Let $X$ be a compact metric space and let $A = PM_r(C(X))P$, where $P \in C(X, M_n)$ is a projection. Let $\Delta : A^1_+ \setminus \{0\} \to (0, 1)$ be an order preserving map. For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $\sigma > 0$, there exists a finite subset $\mathcal{H}_1 \subset A^1_+ \setminus \{0\}$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ and $\delta > 0$ satisfying the following: If $\phi_1, \phi_2 : A \to M_n$ (for some integer $n \geq 1$) are two unital homomorphisms such that

$$\tau \circ \phi_1(h) \geq \Delta(\hat{h}) \quad \text{for all} \quad h \in \mathcal{H}_1$$

and

$$|\tau \circ \phi_1(g) - \tau \circ \phi_2(g)| < \sigma \quad \text{for all} \quad g \in \mathcal{H}_2,$$

then, there exist a projection $p \in M_n$, a unital homomorphism $H : A \to pM_np$,
Lemma 2.4.

Let $X$ be a compact metric space and let $A = PM_r(C(X))P$, where $P \in C(X, M_n)$ is a projection. Let $\Delta : A^q_1 \setminus \{0\} \to (0, 1)$ be an order preserving map. For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $\sigma > 0$, there exists a finite subset $\mathcal{H}_1 \subset A^1_+ \setminus \{0\}$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ and $\delta > 0$ satisfying the following: If $\phi_1, \phi_2 : A \to M_n$ (for some integer $n \geq 1$) are two unital homomorphisms such that

$$\tau \circ \phi_1(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1 \text{ and}$$

$$|\tau \circ \phi_1(g) - \tau \circ \phi_2(g)| < \sigma \text{ for all } g \in \mathcal{H}_2,$$

then, there exist a projection $p \in M_n$, a unital homomorphism $H : A \to pM_np$, unital homomorphisms $h_1, h_2 : A \to (1 - p)M_n(1 - p)$ and a unitary $u \in M_n$ such that
Lemma 2.4.

Let $X$ be a compact metric space and let $A = PM_r(C(X))P$, where $P \in C(X, M_n)$ is a projection. Let $\Delta : A_t^{q, 1} \setminus \{0\} \to (0, 1)$ be an order preserving map. For any $\epsilon > 0$, any finite subset $F \subset A$ and any $\sigma > 0$, there exists a finite subset $\mathcal{H}_1 \subset A_+^{1} \setminus \{0\}$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ and $\delta > 0$ satisfying the following: If $\phi_1, \phi_2 : A \to M_n$ (for some integer $n \geq 1$) are two unital homomorphisms such that

$$\tau \circ \phi_1(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1 \text{ and}$$

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then, there exist a projection $p \in M_n$, a unital homomorphism

$H : A \to pM_n p$, unital homomorphisms $h_1, h_2 : A \to (1 - p)M_n(1 - p)$ and a unitary $u \in M_n$ such that

$$\|\text{Ad} u \circ \phi_1(f) - (h_1(f) + H(f))\| < \epsilon,$$
Lemma 2.4.

Let $X$ be a compact metric space and let $A = PM_r(C(X))P$, where $P \in C(X, M_n)$ is a projection. Let $\Delta : A^{q_1}_1 \setminus \{0\} \to (0, 1)$ be an order preserving map. For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $\sigma > 0$, there exists a finite subset $\mathcal{H}_1 \subset A_+^1 \setminus \{0\}$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ and $\delta > 0$ satisfying the following: If $\phi_1, \phi_2 : A \to M_n$ (for some integer $n \geq 1$) are two unital homomorphisms such that

$$
\tau \circ \phi_1(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1 \text{ and }

|\tau \circ \phi_1(g) - \tau \circ \phi_2(g)| < \sigma \text{ for all } g \in \mathcal{H}_2,
$$

then, there exist a projection $p \in M_n$, a unital homomorphism $H : A \to pM_n p$, unital homomorphisms $h_1, h_2 : A \to (1 - p)M_n(1 - p)$ and a unitary $u \in M_n$ such that

$$
\|\text{Ad } u \circ \phi_1(f) - (h_1(f) + H(f))\| < \epsilon,

\|\phi_2(f) - (h_2(f) + H(f))\| < \epsilon \text{ for all } f \in \mathcal{F}
$$
Lemma 2.4.

Let $X$ be a compact metric space and let $A = \text{PM}_r(C(X))P$, where $P \in C(X, M_n)$ is a projection. Let $\Delta : A^q_1 \setminus \{0\} \to (0, 1)$ be an order preserving map. For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$ and any $\sigma > 0$, there exists a finite subset $\mathcal{H}_1 \subset A^1_+ \setminus \{0\}$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ and $\delta > 0$ satisfying the following: If $\phi_1, \phi_2 : A \to M_n$ (for some integer $n \geq 1$) are two unital homomorphisms such that

$$\tau \circ \phi_1(h) \geq \Delta(\hat{h}) \quad \text{for all } h \in \mathcal{H}_1 \quad \text{and}$$

$$|\tau \circ \phi_1(g) - \tau \circ \phi_2(g)| < \sigma \quad \text{for all } g \in \mathcal{H}_2,$$

then, there exist a projection $p \in M_n$, a unital homomorphism $H : A \to pM_np$, unital homomorphisms $h_1, h_2 : A \to (1 - p)M_n(1 - p)$ and a unitary $u \in M_n$ such that

$$\|\text{Ad} u \circ \phi_1(f) - (h_1(f) + H(f))\| < \epsilon,$$

$$\|\phi_2(f) - (h_2(f) + H(f))\| < \epsilon \quad \text{for all } f \in \mathcal{F}$$

and $\tau(1 - p) < \sigma$,

where $\tau$ is the tracial state of $M_n$. 
Idea of the proof

We may write

\[ \phi_i(f) = K \sum_{k=1}^{K} f(x_k, i) q_k, i \]

for all \( f \in M^r(C(X)) \), where \( \{q_k, i : 1 \leq k \leq K\} \) is a set of mutually orthogonal rank \( r \) projections.

Therefore we may write

\[ \phi_i = \phi_{i,0} \oplus \phi_{i,1} \]

where \( \phi_{i,0} : A \to P_i M_n P_i \) and \( \phi_{i,1} : A \to (1 - P_i) M_n (1 - P_i) \), \( i = 1, 2 \), such that

\[ \text{tr}(P_i) < \sigma \]

and 

\[ [\phi_1, 1] P = [\phi_2, 1] P. \]

We then have

\[ \text{Ad}_u \circ \phi_{1,1} \approx \epsilon/2 \phi_{2,1}. \]
Idea of the proof

We may write

\[ \phi_i(f) = \sum_{k=1}^{K} f(x_{k,i})q_{k,i} \text{ for all } f \in M_r(C(X)), \]

where \( \{q_{k,i} : 1 \leq k \leq K \} \) is a set of mutually orthogonal rank \( r \) projections.

We then have

\[ \text{Ad}u \circ \phi_{1,1} \approx \frac{\epsilon}{2} \phi_{2,1}. \]

Let \( H = \phi_{2,1}. \)
Idea of the proof

We may write

$$\phi_i(f) = \sum_{k=1}^{K} f(x_{k,i})q_{k,i} \quad \text{for all} \quad f \in M_r(C(X)),$$

where \( \{q_{k,i} : 1 \leq k \leq K\} \) is a set of mutually orthogonal rank \( r \) projections.
Idea of the proof

We may write

\[ \phi_i(f) = \sum_{k=1}^{K} f(x_{k,i})q_{k,i} \text{ for all } f \in M_r(C(X)), \]

where \( \{q_{k,i} : 1 \leq k \leq K\} \) is a set of mutually orthogonal rank \( r \) projections. Therefore we may write

\[ \phi_i = \phi_{i,0} \oplus \phi_{i,1}. \]

where \( \phi_{i,0} : A \to P_i M_n P_i \) and \( \phi_{i,1} : A \to (1 - P_i) M_n (1 - P_i), i = 1, 2, \)
Idea of the proof

We may write

\[ \phi_i(f) = \sum_{k=1}^{K} f(x_{k,i})q_{k,i} \text{ for all } f \in M_r(C(X)), \]

where \( \{q_{k,i} : 1 \leq k \leq K\} \) is a set of mutually orthogonal rank \( r \) projections. Therefore we may write

\[ \phi_i = \phi_{i,0} \oplus \phi_{i,1}. \]

where \( \phi_{i,0} : A \to P_iMnP_i \) and \( \phi_{i,1} : A \to (1 - P_i)M_n(1 - P_i), \) \( i = 1, 2, \) such that \( \text{tr}(P_i) < \sigma \) and \( [\phi_{1,1}]|_{\mathcal{P}} = [\phi_{2,1}]|_{\mathcal{P}}. \)
Idea of the proof

We may write

$$\phi_i(f) = \sum_{k=1}^{K} f(x_{k,i})q_{k,i} \text{ for all } f \in M_r(C(X)),$$

where \(\{q_{k,i} : 1 \leq k \leq K\}\) is a set of mutually orthogonal rank \(r\) projections. Therefore we may write

$$\phi_i = \phi_{i,0} \oplus \phi_{i,1}.$$ 

where \(\phi_{i,0} : A \to P_iM_nP_i\) and \(\phi_{i,1} : A \to (1 - P_i)M_n(1 - P_i), i = 1, 2,\) such that \(\text{tr}(P_i) < \sigma\) and \([\phi_{1,1}]_P = [\phi_{2,1}]_P\). We then have

$$\text{Ad} u \circ \phi_{1,1} \approx_{\epsilon/2} \phi_{2,1}.$$ 

Let \(H = \phi_{2,1}\).
Proof: We will only prove the case that $A = M_r(C(X))$. 

Let $\Delta_1 = (1/2)\Delta$. Let $P \in A$ be a finite subset of mutually orthogonal projections, $H_1' \subset A_1 + \{0\}$ (in place of $H_1$) be a finite subset, $H_2' \subset A_s$ (in place of $H_2$) be a finite subset and $\delta_1 > 0$ (in place of $\delta$) required by Theorem 2.3 for $\epsilon/2$ (in place of $\epsilon$), $F$ and $\Delta_1$. Without loss of generality, we may assume that $1_A \in F$, $1_A \in H_1' \subset H_2'$ and $H_2' \subset A_1 + \{0\}$. Put $\sigma_0 = \min \{\Delta(\hat{g}) : g \in H_2'\}$.

We may write $P = \{p_1, p_2, \ldots, p_{k_1}\}$. Without loss of generality, we may assume that $\{p_i : 1 \leq i \leq k_1\}$ is a set of mutually orthogonal projections such that $1_A = \sum_{i=1}^{k_1} p_i$. Let $H_1 = H_1' \cup \{p_i : 1 \leq i \leq k_1\} \cup H_1''$ and $H_2 = H_2' \cup H_1$. Let $\sigma_1 = \min \{\Delta(\hat{g}) : g \in H_2\}$. Choose $\delta = \min \{\sigma_0 \cdot \sigma/4, \sigma_0 \cdot \delta_1/4, \sigma_1/16 \}$.

Suppose now that $\phi_1, \phi_2 : A \to M_n$ are two unital homomorphisms described in the lemma for the above $H_1, H_2$ and $\Delta$.
Proof: We will only prove the case that \( A = M_r(C(X)) \). Let \( \Delta_1 = (1/2)\Delta \).
Proof: We will only prove the case that \( A = M_r(C(X)) \). Let \( \Delta_1 = (1/2)\Delta \). Let \( \mathcal{P} \in A \) be a finite subset of mutually orthogonal projections,
Proof: We will only prove the case that $A = M_r(C(X))$. Let $\Delta_1 = (1/2)\Delta$. Let $\mathcal{P} \in A$ be a finite subset of mutually orthogonal projections, $\mathcal{H}_1' \subset A^1_+ \setminus \{0\}$ (in place of $\mathcal{H}_1$) be a finite subset,
Proof : We will only prove the case that \( A = M_r(C(X)) \). Let \( \Delta_1 = (1/2)\Delta \). Let \( \mathcal{P} \in A \) be a finite subset of mutually orthogonal projections, \( \mathcal{H}_1' \subset A_+^1 \setminus \{0\} \) (in place of \( \mathcal{H}_1 \)) be a finite subset, \( \mathcal{H}_2' \subset A_{s.a.} \) (in place of \( \mathcal{H}_2 \)) be a finite subset.
Proof: We will only prove the case that \( A = M_r(C(X)) \). Let \( \Delta_1 = (1/2)\Delta \). Let \( \mathcal{P} \in A \) be a finite subset of mutually orthogonal projections, \( \mathcal{H}_1' \subset A_+^1 \setminus \{0\} \) (in place of \( \mathcal{H}_1 \)) be a finite subset, \( \mathcal{H}_2' \subset A_{s.a.} \) (in place of \( \mathcal{H}_2 \)) be a finite subset and \( \delta_1 > 0 \) (in place of \( \delta \)) required by Theorem 2.3 for \( \epsilon/2 \) (in place of \( \epsilon \), \( \mathcal{F} \) and \( \Delta_1 \).
Proof : We will only prove the case that $A = M_r(C(X))$. Let $\Delta_1 = (1/2)\Delta$. Let $\mathcal{P} \in A$ be a finite subset of mutually orthogonal projections, $\mathcal{H}_1' \subset A_+^1 \setminus \{0\}$ (in place of $\mathcal{H}_1$) be a finite subset, $\mathcal{H}_2' \subset A_{s.a.}$ (in place of $\mathcal{H}_2$) be a finite subset and $\delta_1 > 0$ (in place of $\delta$) required by Theorem 2.3 for $\epsilon/2$ (in place of $\epsilon$), $\mathcal{F}$ and $\Delta_1$. Without loss of generality, we may assume that $1_A \in \mathcal{F}$, $1_A \in \mathcal{H}_1' \subset \mathcal{H}_2'$ and $\mathcal{H}_2' \subset A_+^1 \setminus \{0\}$.
Proof: We will only prove the case that $A = M_r(C(X))$. Let $\Delta_1 = (1/2)\Delta$. Let $\mathcal{P} \in A$ be a finite subset of mutually orthogonal projections, $\mathcal{H}'_1 \subset A^1_+ \setminus \{0\}$ (in place of $\mathcal{H}_1$) be a finite subset, $\mathcal{H}'_2 \subset A_{s.a.}$ (in place of $\mathcal{H}_2$) be a finite subset and $\delta_1 > 0$ (in place of $\delta$) required by Theorem 2.3 for $\epsilon/2$ (in place of $\epsilon$), $\mathcal{F}$ and $\Delta_1$. Without loss of generality, we may assume that $1_A \in \mathcal{F}$, $1_A \in \mathcal{H}'_1 \subset \mathcal{H}'_2$ and $\mathcal{H}'_2 \subset A^1_+ \setminus \{0\}$. Put

$$\sigma_0 = \min \{ \Delta_1(\hat{g}) : g \in \mathcal{H}'_2 \}. \quad (e10.50)$$
Proof: We will only prove the case that $A = M_r(C(X))$. Let
$\Delta_1 = (1/2)\Delta$. Let $\mathcal{P} \in A$ be a finite subset of mutually orthogonal
projections, $\mathcal{H}'_1 \subset A^1_+ \setminus \{0\}$ (in place of $\mathcal{H}_1$) be a finite subset, $\mathcal{H}'_2 \subset A_{s.a.}$
(in place of $\mathcal{H}_2$) be a finite subset and $\delta_1 > 0$ (in place of $\delta$) required by
Theorem 2.3 for $\epsilon/2$ (in place of $\epsilon$), $\mathcal{F}$ and $\Delta_1$. Without loss of generality,
we may assume that $1_A \in \mathcal{F}$, $1_A \in \mathcal{H}'_1 \subset \mathcal{H}'_2$ and $\mathcal{H}'_2 \subset A^1_+ \setminus \{0\}$. Put

$$\sigma_0 = \min\{\Delta_1(\hat{g}) : g \in \mathcal{H}'_2\}. \tag{e10.50}$$

We may write $\mathcal{P} = \{p_1, p_2, ..., p_{k_1}\}$. 
**Proof:** We will only prove the case that $A = M_r(C(X))$. Let $\Delta_1 = (1/2)\Delta$. Let $\mathcal{P} \in A$ be a finite subset of mutually orthogonal projections, $\mathcal{H}'_1 \subset A^1_+ \setminus \{0\}$ (in place of $\mathcal{H}_1$) be a finite subset, $\mathcal{H}'_2 \subset A_{s.a.}$ (in place of $\mathcal{H}_2$) be a finite subset and $\delta_1 > 0$ (in place of $\delta$) required by Theorem 2.3 for $\epsilon/2$ (in place of $\epsilon$), $\mathcal{F}$ and $\Delta_1$. Without loss of generality, we may assume that $1_A \in \mathcal{F}$, $1_A \in \mathcal{H}'_1 \subset \mathcal{H}'_2$ and $\mathcal{H}'_2 \subset A^1_+ \setminus \{0\}$. Put

$$\sigma_0 = \min \{ \Delta_1(\hat{g}) : g \in \mathcal{H}'_2 \}.$$  \hspace{1cm} (e10.50)

We may write $\mathcal{P} = \{p_1, p_2, \ldots, p_{k_1}\}$. Without loss of generality, we may assume that $\{p_i : 1 \leq i \leq k_1\}$ is a set of mutually orthogonal projections such that $1_A = \sum_{i=1}^{k_1} p_i$. 


**Proof**: We will only prove the case that \( A = M_r(C(X)) \). Let \( \Delta_1 = (1/2)\Delta \). Let \( P \in A \) be a finite subset of mutually orthogonal projections, \( H'_1 \subset A^1_+ \setminus \{0\} \) (in place of \( H_1 \)) be a finite subset, \( H'_2 \subset A_{s.a.} \) (in place of \( H_2 \)) be a finite subset and \( \delta_1 > 0 \) (in place of \( \delta \)) required by Theorem 2.3 for \( \epsilon/2 \) (in place of \( \epsilon \)), \( F \) and \( \Delta_1 \). Without loss of generality, we may assume that \( 1_A \in F \), \( 1_A \in H'_1 \subset H'_2 \) and \( H'_2 \subset A^1_+ \setminus \{0\} \). Put

\[
\sigma_0 = \min\{\Delta_1(\hat{g}) : g \in H'_2\}. \quad (e\ 10.50)
\]

We may write \( P = \{p_1, p_2, \ldots, p_{k_1}\} \). Without loss of generality, we may assume that \( \{p_i : 1 \leq i \leq k_1\} \) is a set of mutually orthogonal projections such that \( 1_A = \sum_{i=1}^{k_1} p_i \). Let \( H_1 = H'_1 \cup \{p_i : 1 \leq i \leq k_1\} \cup H''_1 \) and \( H_2 = H'_2 \cup H'_1 \).
Proof: We will only prove the case that $A = M_r(C(X))$. Let $\Delta_1 = (1/2)\Delta$. Let $\mathcal{P} \in A$ be a finite subset of mutually orthogonal projections, $\mathcal{H}'_1 \subset A^1_+ \setminus \{0\}$ (in place of $\mathcal{H}_1$) be a finite subset, $\mathcal{H}'_2 \subset A_{s.a.}$ (in place of $\mathcal{H}_2$) be a finite subset and $\delta_1 > 0$ (in place of $\delta$) required by Theorem 2.3 for $\epsilon/2$ (in place of $\epsilon$), $\mathcal{F}$ and $\Delta_1$. Without loss of generality, we may assume that $1_A \in \mathcal{F}$, $1_A \in \mathcal{H}'_1 \subset \mathcal{H}'_2$ and $\mathcal{H}'_2 \subset A^1_+ \setminus \{0\}$. Put

$$\sigma_0 = \min\{\Delta_1(\hat{g}) : g \in \mathcal{H}'_2\}. \quad (e10.50)$$

We may write $\mathcal{P} = \{p_1, p_2, \ldots, p_{k_1}\}$. Without loss of generality, we may assume that $\{p_i : 1 \leq i \leq k_1\}$ is a set of mutually orthogonal projections such that $1_A = \sum_{i=1}^{k_1} p_i$. Let $\mathcal{H}_1 = \mathcal{H}'_1 \cup \{p_i : 1 \leq i \leq k_1\} \cup \mathcal{H}''_1$ and $\mathcal{H}_2 = \mathcal{H}'_2 \cup \mathcal{H}_1$. Let $\sigma_1 = \min\{\Delta(\hat{g}) : g \in \mathcal{H}_2\}$. 
Proof: We will only prove the case that \( A = M_r(C(X)) \). Let \( \Delta_1 = (1/2)\Delta \). Let \( \mathcal{P} \in A \) be a finite subset of mutually orthogonal projections, \( \mathcal{H}'_1 \subset A_+^1 \setminus \{0\} \) (in place of \( \mathcal{H}_1 \)) be a finite subset, \( \mathcal{H}'_2 \subset A_{s.a.} \) (in place of \( \mathcal{H}_2 \)) be a finite subset and \( \delta_1 > 0 \) (in place of \( \delta \)) required by Theorem 2.3 for \( \epsilon/2 \) (in place of \( \epsilon \)), \( \mathcal{F} \) and \( \Delta_1 \). Without loss of generality, we may assume that \( 1_A \in \mathcal{F} \), \( 1_A \in \mathcal{H}'_1 \subset \mathcal{H}'_2 \) and \( \mathcal{H}'_2 \subset A_+^1 \setminus \{0\} \). Put

\[
\sigma_0 = \min\{\Delta_1(\hat{g}) : g \in \mathcal{H}'_2\}. \tag{e10.50}
\]

We may write \( \mathcal{P} = \{p_1, p_2, \ldots, p_{k_1}\} \). Without loss of generality, we may assume that \( \{p_i : 1 \leq i \leq k_1\} \) is a set of mutually orthogonal projections such that \( 1_A = \sum_{i=1}^{k_1} p_i \). Let \( \mathcal{H}_1 = \mathcal{H}'_1 \cup \{p_i : 1 \leq i \leq k_1\} \cup \mathcal{H}''_1 \) and \( \mathcal{H}_2 = \mathcal{H}'_2 \cup \mathcal{H}_1 \). Let \( \sigma_1 = \min\{\Delta(\hat{g}) : g \in \mathcal{H}_2\} \). Choose \( \delta = \min\{\sigma_0 \cdot \sigma/4k_1, \sigma_0 \cdot \delta_1/4k_1, \sigma_1/16k_1\} \).
Proof: We will only prove the case that $A = M_r(C(X))$. Let $\Delta_1 = (1/2)\Delta$. Let $\mathcal{P} \in A$ be a finite subset of mutually orthogonal projections, $\mathcal{H}'_1 \subset A_1^+ \setminus \{0\}$ (in place of $\mathcal{H}_1$) be a finite subset, $\mathcal{H}'_2 \subset A_{s.a.}$ (in place of $\mathcal{H}_2$) be a finite subset and $\delta_1 > 0$ (in place of $\delta$) required by Theorem 2.3 for $\epsilon/2$ (in place of $\epsilon$), $\mathcal{F}$ and $\Delta_1$. Without loss of generality, we may assume that $1_A \in \mathcal{F}$, $1_A \in \mathcal{H}'_1 \subset \mathcal{H}'_2$ and $\mathcal{H}'_2 \subset A_1^+ \setminus \{0\}$. Put

$$\sigma_0 = \min \{\Delta_1(\hat{g}) : g \in \mathcal{H}'_2\}. \quad (e10.50)$$

We may write $\mathcal{P} = \{p_1, p_2, \ldots, p_{k_1}\}$. Without loss of generality, we may assume that $\{p_i : 1 \leq i \leq k_1\}$ is a set of mutually orthogonal projections such that $1_A = \sum_{i=1}^{k_1} p_i$. Let $\mathcal{H}_1 = \mathcal{H}'_1 \cup \{p_i : 1 \leq i \leq k_1\} \cup \mathcal{H}_1''$ and $\mathcal{H}_2 = \mathcal{H}'_2 \cup \mathcal{H}_1$. Let $\sigma_1 = \min \{\Delta(\hat{g}) : g \in \mathcal{H}_2\}$. Choose $\delta = \min \{\sigma_0 \cdot \sigma/4k_1, \sigma_0 \cdot \delta_1/4k_1, \sigma_1/16k_1\}$. Suppose now that $\phi_1, \phi_2 : A \to M_n$ are two unital homomorphisms
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Proof: We will only prove the case that \( A = M_r(C(X)) \). Let \( \Delta_1 = (1/2)\Delta \). Let \( \mathcal{P} \in A \) be a finite subset of mutually orthogonal projections, \( \mathcal{H}'_1 \subset A^1_+ \setminus \{0\} \) (in place of \( \mathcal{H}_1 \)) be a finite subset, \( \mathcal{H}'_2 \subset A_{s.a.} \) (in place of \( \mathcal{H}_2 \)) be a finite subset and \( \delta_1 > 0 \) (in place of \( \delta \)) required by Theorem 2.3 for \( \epsilon/2 \) (in place of \( \epsilon \)), \( \mathcal{F} \) and \( \Delta_1 \). Without loss of generality, we may assume that \( 1_A \in \mathcal{F} \), \( 1_A \in \mathcal{H}'_1 \subset \mathcal{H}'_2 \) and \( \mathcal{H}'_2 \subset A^1_+ \setminus \{0\} \). Put

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We may write \( \phi_j(f) = \sum_{k=1}^{n} f(x_{k,j})q_{k,j} \) for all \( f \in M_r(C(X)) \), where \( \{ q_{k,j} : 1 \leq k \leq n \} \) (\( j = 1, 2 \)) is a set of mutually orthogonal rank \( r \) projections and \( x_{k,j} \in X \).
We have

\[ |\tau \circ \phi_1(p_i) - \tau \circ \phi_2(p_i)| < \delta, \quad i = 1, 2, \ldots, k_1, \]  

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where $\tau$ is the tracial state on $M_n$. 


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$$\tau(P_{0,j}) < k_1 \delta < \sigma_0 \cdot \sigma, \quad j = 1, 2,$$

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$$\text{rank}(P_{0,1}) = \text{rank}(P_{0,2}),$$

and unital homomorphisms $\phi_1, \phi_2: A \to P_{0,1}M_n P_{0,1}, \phi_1, \phi_2: A \to P_{0,2}M_n P_{0,2}, \phi_1, \phi_2: A \to (1 - P_{0,1})M_n(1 - P_{0,1}), \phi_1, \phi_2: A \to (1 - P_{0,2})M_n(1 - P_{0,2})$. 

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We have
\[
|\tau \circ \phi_1(p_i) - \tau \circ \phi_2(p_i)| < \delta, \quad i = 1, 2, \ldots, k_1, \quad (e10.51)
\]

where \(\tau\) is the tracial state on \(M_n\). Therefore, there exists a projection \(P_{0,j} \in M_n\) such that
\[
\tau(P_{0,j}) < k_1 \delta < \sigma_0 \cdot \sigma, \quad j = 1, 2, \quad (e10.52)
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\[\phi_{2,0} : A \to P_{0,2}M_nP_{0,2}, \phi_{1,1} : A \to (1 - P_{0,1})M_n(1 - P_{0,1}) \text{ and} \]
\[\phi_{1,2} : A \to (1 - P_{0,2})M_n(1 - P_{0,2}) \text{ such that} \]
\[
\phi_1 = \phi_{1,0} \oplus \phi_{1,1}, \quad \phi_2 = \phi_{2,0} \oplus \phi_{2,1}, \quad (e10.53)
\]
\[
\tau \circ \phi_{1,1}(p_i) = \tau \circ \phi_{1,2}(p_i), \quad i = 1, 2, \ldots, k_1. \quad (e10.54)
\]
By replacing $\phi_1$ by $\text{Ad} \circ \phi_1$, simplifying the notation, without loss of generality, we may assume that $P_{0,1} = P_{0,2}$. It follows (see ??) that

$$[\phi_{1,1}]_p = [\phi_{2,1}]_p. \quad (e\ 10.55)$$

By (e\ 10.52) and choice of $\sigma_0$, we also have

$$\tau \circ \phi_{1,1}(g) \geq \Delta_1(\hat{g}) \text{ for all } g \in \mathcal{H}'_1 \text{ and } \quad (e\ 10.56)$$

$$|\tau \circ \phi_{1,1}(g) - \tau \circ \phi_{1,2}(g)| < \sigma_0 \cdot \delta_1 \text{ for all } g \in \mathcal{H}'_2. \quad (e\ 10.57)$$

Therefore

$$t \circ \phi_{1,1}(g) \geq \Delta_1(\hat{g}) \text{ for all } g \in \mathcal{H}'_1 \text{ and } \quad (e\ 10.58)$$

$$|t \circ \phi_{1,1}(g) - t \circ \phi_{1,2}(g)| < \delta_1 \text{ for all } g \in \mathcal{H}'_2, \quad (e\ 10.59)$$

where $t$ is the tracial state on $(1 - P_{1,0})M_n(1 - P_{1,0})$. By applying ??, there exists a unitary $\nu_1 \in (1 - P_{1,0})M_n(1 - P_{1,0})$ such that

$$\|\text{Ad} \circ \phi_{1,1}(f) - \phi_{2,1}(f)\| < \epsilon/16 \text{ for all } f \in \mathcal{F}. \quad (e\ 10.60)$$

Put $H = \phi_{2,1}$ and $p = P_{1,0}$. The lemma for the case that $A = M_r(C(X))$ follows.
Corollary 2.5.

Let $X$ be a compact metric space and let $A = PC(X, M_n)P$, where $P \in C(X, F)$ is a projection. Let $\Delta : A^{\mathbb{Q}}_{\{0\}} \to (0, 1)$ be an order-preserving map and let $\alpha > 1/2$. For any $\epsilon > 0$, any finite subset $F \subset A$, any finite subset $H_0 \subset A_{\{0\}}$, and any integer $K \geq 1$. There is an integer $N \geq 1$, a finite subset $H_1 \subset A_{\{0\}}$, a finite subset $H_2 \subset A$, $\delta > 0$ satisfying the following:

If $\phi_1, \phi_2 : A \to M_n$ (for any integer $n \geq N$) are two unital homomorphisms such that $\tau \circ \phi_1(h) \geq \Delta(\hat{h})$ for all $h \in H_1$ and $|\tau \circ \phi_1(g) - \tau \circ \phi_2(g)| < \delta$ for all $g \in H_2$, then, there exists a unitary $u \in M_n$ such that $Huaxin Lin$
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Let $X$ be a compact metric space and let $A = PC(X, M_n)P$, where $P \in C(X, F)$ is a projection.
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Let $X$ be a compact metric space and let $A = PC(X, M_n)P$, where $P \in C(X, F)$ is a projection. Let $\Delta : A_{q,1}^+ \setminus \{0\} \to (0, 1)$ be an order preserving map and let $1 > \alpha > 1/2$. For any $\epsilon > 0$, any finite subset $F \subset A$, any finite subset $H \subset A_{1}^+ \setminus \{0\}$ and any integer $K \geq 1$, there is an integer $N \geq 1$, a finite subset $H_1 \subset A_{1}^+ \setminus \{0\}$, a finite subset $H_2 \subset A_s$, a $\delta > 0$ satisfying the following:

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Basic Homotopy Lemmas Introduction

June 8th, 2015, RMMC/CBMS University of Wyoming
Corollary 2.5.

Let $X$ be a compact metric space and let $A = PC(X, M_n)P$, where $P \in C(X, F)$ is a projection. Let $\Delta : A^{q,1}_+ \setminus \{0\} \rightarrow (0, 1)$ be an order preserving map and let $1 > \alpha > 1/2$.

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For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$, any finite subset $\mathcal{H}_0 \subset A^1_+ \setminus \{0\}$ and any integer $K \geq 1$. There is an integer $N \geq 1$, a finite subset $\mathcal{H}_1 \subset A^1_+ \setminus \{0\}$,
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For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$, any finite subset $\mathcal{H}_0 \subset A^1_+ \setminus \{0\}$ and any integer $K \geq 1$. There is an integer $N \geq 1$, a finite subset $\mathcal{H}_1 \subset A^1_+ \setminus \{0\}$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$, $\delta > 0$ satisfying the following:
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Let $X$ be a compact metric space and let $A = \text{PC}(X, M_n)P$, where $P \in C(X, F)$ is a projection. Let $\Delta : A^{q,1}_+ \setminus \{0\} \to (0, 1)$ be an order preserving map and let $1 > \alpha > 1/2$.

For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$, any finite subset $\mathcal{H}_0 \subset A^{1}_+ \setminus \{0\}$ and any integer $K \geq 1$. There is an integer $N \geq 1$, a finite subset $\mathcal{H}_1 \subset A^{1}_+ \setminus \{0\}$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$, $\delta > 0$ satisfying the following: If $\phi_1, \phi_2 : A \to M_n$ (for any integer $n \geq N$) are two unital homomorphisms
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For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$, any finite subset $\mathcal{H}_0 \subset A^1_+ \backslash \{0\}$ and any integer $K \geq 1$. There is an integer $N \geq 1$, a finite subset $\mathcal{H}_1 \subset A^1_+ \backslash \{0\}$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$, $\delta > 0$ satisfying the following:

If $\phi_1, \phi_2 : A \to M_n$ (for any integer $n \geq N$) are two unital homomorphisms such that

$$\tau \circ \phi_1(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1$$

and
Corollary 2.5.

Let $X$ be a compact metric space and let $A = PC(X, M_n)P$, where $P \in C(X, F)$ is a projection. Let $\Delta : A_{+1}^q \setminus \{0\} \to (0, 1)$ be an order preserving map and let $1 > \alpha > 1/2$.

For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$, any finite subset $\mathcal{H}_0 \subset A_{+1}^1 \setminus \{0\}$ and any integer $K \geq 1$. There is an integer $N \geq 1$, a finite subset $\mathcal{H}_1 \subset A_{+1}^1 \setminus \{0\}$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$, $\delta > 0$ satisfying the following: If $\phi_1, \phi_2 : A \to M_n$ (for any integer $n \geq N$) are two unital homomorphisms such that

$$
\tau \circ \phi_1(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1 \text{ and } \\
|\tau \circ \phi_1(g) - \tau \circ \phi_2(g)| < \delta \text{ for all } g \in \mathcal{H}_2,$$

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Corollary 2.5.
Let $X$ be a compact metric space and let $A = PC(X, M_n)P$, where $P \in C(X, F)$ is a projection. Let $\Delta : A^{q, 1}_+ \setminus \{0\} \to (0, 1)$ be an order preserving map and let $1 > \alpha > 1/2$.
For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$, any finite subset $\mathcal{H}_0 \subset A^{1}_+ \setminus \{0\}$ and any integer $K \geq 1$. There is an integer $N \geq 1$, a finite subset $\mathcal{H}_1 \subset A^{1}_+ \setminus \{0\}$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$, $\delta > 0$ satisfying the following: If $\phi_1, \phi_2 : A \to M_n$ (for any integer $n \geq N$) are two unital homomorphisms such that

$$
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$$

$$
|\tau \circ \phi_1(g) - \tau \circ \phi_2(g)| < \delta \text{ for all } g \in \mathcal{H}_2,
$$

then, there exists a unitary $u \in M_n$ such that
\[ \| \text{Ad } u \circ \phi_1(f) - (h_1(f) + \text{diag}(\psi(f), \psi(f), ..., \psi(f))) \| < \epsilon, \]

\[ \tau \circ \psi(g) \geq \alpha \Delta(\hat{g})K \text{ for all } g \in H_0, \]

\[ \tau \in T(M_n), h_1, h_2 : A \to e_0 M_n e_0, \psi : A \to e_1 M_n e_1 \text{ are unital homomorphisms,} \]

\[ e_0, e_1, e_2, ..., e_K \in M_n \text{ are mutually orthogonal non-zero projections,} \]

\[ e_1, e_2, ..., e_K \text{ are equivalent, } e_0 \precsim e_1 \text{ and } e_0 + \sum_{i=1}^{K} e_i = 1 M_n. \]

Remark: If \( X \) has infinitely many points, then there is no need to mention the integer \( N \). The integer \( n \) will be large when \( H_0 \) is large.
\[ \| \text{Ad} u \circ \phi_1(f) - (h_1(f) + \text{diag}(\psi(f), \psi(f), \ldots, \psi(f))) \| < \epsilon, \]

\[ \| \phi_2(f) - (h_2(f) + \text{diag}(\psi(f), \psi(f), \ldots, \psi(f))) \| < \epsilon \text{ for all } f \in \mathcal{F}, \]
\[ \| \text{Ad} \, u \circ \phi_1(f) - (h_1(f) + \text{diag}(\psi(f), \psi(f), \ldots, \psi(f))) \| < \epsilon, \]
\[ \| \phi_2(f) - (h_2(f) + \text{diag}(\psi(f), \psi(f), \ldots, \psi(f))) \| < \epsilon \text{ for all } f \in \mathcal{F}, \]
\[ \text{and } \tau \circ \psi(g) \geq \alpha \frac{\Delta(\hat{g})}{K} \text{ for all } g \in \mathcal{H}_0, \]
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\[ \| \phi_2 (f) - (h_2 (f) + \text{diag}(\psi(f), \psi(f), \ldots, \psi(f))) \| < \epsilon \quad \text{for all } f \in \mathcal{F}, \]

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\[ \tau \in T(M_n), \ h_1, h_2 : A \to e_0 M_n e_0, \]
\[ \| \text{Ad} u \circ \phi_1(f) - (h_1(f) + \text{diag}(\psi(f), \psi(f), ..., \psi(f))) \| < \epsilon, \]
\[ \| \phi_2(f) - (h_2(f) + \text{diag}(\psi(f), \psi(f), ..., \psi(f))) \| < \epsilon \text{ for all } f \in \mathcal{F}, \]
\[ \text{and } \tau \circ \psi(g) \geq \alpha \frac{\Delta(\hat{g})}{K} \text{ for all } g \in \mathcal{H}_0, \]

\( \tau \in T(M_n), \ h_1, h_2 : A \rightarrow e_0 M_n e_0, \ \psi : A \rightarrow e_1 M_n e_1 \) are unital homomorphisms,
\[ \| \text{Ad} \ u \circ \phi_1(f) - (h_1(f) + \text{diag}(\psi(f), \psi(f), \ldots, \psi(f))) \| < \epsilon, \]

\[ \| \phi_2(f) - (h_2(f) + \text{diag}(\psi(f), \psi(f), \ldots, \psi(f))) \| < \epsilon \quad \text{for all } f \in \mathcal{F}, \]

and \[ \tau \circ \psi(g) \geq \alpha \frac{\Delta(\hat{g})}{K} \quad \text{for all } g \in \mathcal{H}_0, \]

\( \tau \in T(M_n), \ h_1, h_2 : A \to e_0 M_n e_0, \ \psi : A \to e_1 M_n e_1 \) are unital homomorphisms, \( e_0, e_1, e_2, \ldots, e_K \in M_n \) are mutually orthogonal non-zero projections,
\[ \| \text{Ad } u \circ \phi_1(f) - (h_1(f) + \text{diag}(\psi(f), \psi(f), ..., \psi(f))) \| < \epsilon, \]
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\begin{align*}
&\|\text{Ad } u \circ \phi_1(f) - (h_1(f) + \text{diag}(\psi(f), \psi(f), \ldots, \psi(f)))\| < \epsilon, \\
&\|\phi_2(f) - (h_2(f) + \text{diag}(\psi(f), \psi(f), \ldots, \psi(f)))\| < \epsilon \text{ for all } f \in \mathcal{F}, \\
&\text{and } \tau \circ \psi(g) \geq \alpha \frac{\Delta(\hat{g})}{K} \text{ for all } g \in \mathcal{H}_0,
\end{align*}

\tau \in T(M_n), \ h_1, h_2 : A \to e_0 M_n e_0, \ \psi : A \to e_1 M_n e_1 \text{ are unital homomorphisms, } e_0, e_1, e_2, \ldots, e_K \in M_n \text{ are mutually orthogonal non-zero projections, } e_1, e_2, \ldots, e_K \text{ are equivalent, } e_0 \lesssim e_1 \text{ and } e_0 + \sum_{i=1}^{K} e_i = 1_{M_n}.

Remark: If $X$ has infinitely many points, then there is no need to mention the integer $N$. 
\[ \| \text{Ad} u \circ \phi_1(f) - (h_1(f) + \text{diag}(\psi(f), \psi(f), \ldots, \psi(f))) \| < \epsilon, \]
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\( \tau \in T(M_n) \), \( h_1, h_2 : A \to e_0 M_n e_0 \), \( \psi : A \to e_1 M_n e_1 \) are unital homomorphisms, \( e_0, e_1, e_2, \ldots, e_K \in M_n \) are mutually orthogonal non-zero projections, \( e_1, e_2, \ldots, e_K \) are equivalent, \( e_0 \preceq e_1 \) and \( e_0 + \sum_{i=1}^{K} e_i = 1_{M_n} \).

Remark: If \( X \) has infinitely many points, then there is no need to mention the integer \( N \). The integer \( n \) will be large when \( \mathcal{H}_0 \) is large.
Idea of the proof
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We can write

$$H \approx _{\epsilon} \phi + \text{diag}(\psi, \psi, \cdots, \psi).$$
Proof: We prove the case that $C = C(X)$. 
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Let \( X, F, P, A \) and \( \alpha \) be as in the corollary. Let \( \epsilon > 0 \), let \( F \subset A \) be a finite subset, let \( \mathcal{H}_0 \subset A_+^1 \backslash \{0\} \) and let \( K \geq 1 \).

There is an integer \( N \geq 1 \), a finite subset \( \mathcal{H}_1 \subset A_+^1 \backslash \{0\} \) satisfying the following: Suppose that \( H : A \to M_n \) (for some \( n \geq N \)) is a unital homomorphism such that

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\tau \circ H(g) \geq \Delta(\hat{g}) \text{ for all } g \in \mathcal{H}_0.
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Then there are mutually orthogonal projections $e_0, e_1, e_2, \ldots, e_K \in M_n$, a unital homomorphism $\phi : A \to e_0M_ne_0$ and a unital homomorphism $\psi : A \to e_1M_ne_1$.
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\[
\| H(f) - (\phi(f) \oplus \text{diag}(\psi(f), \psi(f), \ldots, \psi(f)))\| < \epsilon \text{ for all } f \in F,
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$$\|H(f) - (\phi(f) \oplus \text{diag}(\psi(f), \psi(f), \ldots, \psi(f)))\| < \epsilon \text{ for all } f \in \mathcal{F},$$

$$\tau \circ \psi(g) \geq \alpha \Delta(\hat{g}) \text{ for all } g \in \mathcal{H}_0.$$
\[ \sigma_0 = \left(1 - \alpha \right) \frac{1}{4} \min \{ \Delta(\hat{g}) : g \in H_0 \} > 0. \] (e 10.62)

Let \( \epsilon_1 = \min \{ \epsilon/16, \sigma_0 \} \) and let \( F_1 = F \cup H_0 \).

Choose \( d_0 > 0 \) such that
\[ |f(x) - f(x')| < \epsilon_1 \] for all \( f \in F_1 \), provided that \( x, x' \in X \) and \( \text{dist}(x, x') < d_0 \).

Choose \( \xi_1, \xi_2, \ldots, \xi_m \in X \) such that
\[ \bigcup_{j=1}^{m} B(\xi_j, d_0/2) \supset X, \] where \( B(\xi, r) = \{ x \in X : \text{dist}(x, \xi) < r \} \).

There is \( d_1 > 0 \) such that \( d_1 < d_0/2 \) and
\[ B(\xi_j, d_1) \cap B(\xi_i, d_1) = \emptyset \] (e 10.64) if \( i \neq j \).

There is, for each \( j \), a function \( h_j \in C(X) \) with \( 0 \leq h_j \leq 1 \),
\[ h_j(x) = 1 \] if \( x \in B(\xi_j, d_1/2) \) and \( h_j(x) = 0 \) if \( x \not\in B(\xi_j, d_1) \).

Define \( H_1 = H_0 \cup \{ h_j : 1 \leq j \leq m \} \) and put \( \sigma_1 = \min \{ \Delta(\hat{g}) : g \in H_1 \} \).

Choose an integer \( N_0 \geq 1 \) such that \( \frac{1}{N_0} < \sigma_1 \cdot \left(1 - \alpha \right) / 4 \) and
\[ N = 4^m (N_0 + 1) 2^{(K+1)2}. \]
Put 

\[ \sigma_0 = \left( \frac{(1 - \alpha)}{4} \right) \min \{ \Delta(\hat{g}) : g \in \mathcal{H}_0 \} > 0. \]  

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Choose \( \xi_1, \xi_2, \ldots, \xi_m \in X \) such that \( \bigcup_{j=1}^m B(\xi_j, d_0/2) \supset X \), where

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\[
B(\xi_j, d_1) \cap B(\xi_i, d_1) = \emptyset \tag{e 10.64}
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\[ B(\xi_j, d_1) \cap B(\xi_i, d_1) = \emptyset \] (e10.64)
if \( i \neq j \). There is, for each \( j \), a function \( h_j \in C(X) \) with \( 0 \leq h_j \leq 1 \),
\[ h_j(x) = 1 \text{ if } x \in B(\xi_j, d_1/2) \text{ and } h_j(x) = 0 \text{ if } x \not\in B(\xi_j, d_1). \]
Define \( \mathcal{H}_1 = \mathcal{H}_0 \cup \{ h_j : 1 \leq j \leq m \} \) and put
\[ \sigma_1 = \min \{ \Delta(\hat{g}) : g \in \mathcal{H}_1 \}. \] (e10.65)

Choose an integer \( N_0 \geq 1 \) such that \( 1/N_0 < \sigma_1 \cdot (1 - \alpha)/4 \) and
\[ N = 4m(N_0 + 1)^2(K + 1)^2. \]

Now let \( H : C(X) \to M_n \) be a unital homomorphism with \( n > N \).
Let $Y_1 = B(\xi_1, d_0/2) \setminus \bigcup_{i=2}^m B(\xi_i, d_1)$,

$Y_2 = B(\xi_2, d_0/2) \setminus (Y_1 \cup \bigcup_{i=3}^m B(\xi_i, d_1))$,

$Y_j = B(\xi_j, d_0/2) \setminus (\bigcup_{i=1}^{j-1} Y_i \cup \bigcup_{i=j+1}^m B(\xi_i, d_1))$, $j = 1, 2, \ldots, m$. Note that $Y_j \cap Y_i = \emptyset$ if $i \neq j$ and $B(\xi_j, d_1) \subset Y_j$. We write that

$$H(f) = \sum_{i=1}^n f(x_i)p_i = \sum_{j=1}^m \left( \sum_{x_i \in Y_j} f(x_i)p_i \right) \text{ for all } f \in C(X), \quad (e\,10.66)$$

where $\{p_1, p_2, \ldots, p_n\}$ is a set of mutually orthogonal rank one projections in $M_n$, $\{x_1, x_2, \ldots, x_n\} \subset X$. Let $R_j$ be the cardinality of $\{x_i : x_i \in Y_j\}$. Then, by (e\,10.61),

$$R_j \geq N\tau \circ H(h_j) \geq N\Delta(\hat{h}_j) \geq (N_0 + 1)^2 K\sigma_1 \geq (N_0 + 1)K^2, \quad (e\,10.67)$$

$j = 1, 2, \ldots, m$. Write $R_j = S_j K + r_j$, where $S_j \geq N_0 Km$ and $0 \leq r_j < K$, $j = 1, 2, \ldots, m$. Choose $x_{j,1}, x_{j,2}, \ldots, x_{j,r_j} \subset \{x_i \in Y_j\}$ and denote $Z_j = \{x_{j,1}, x_{j,2}, \ldots, x_{j,r_j}\}$, $j = 1, 2, \ldots, m$. 


Therefore we may write

\[
H(f) = \sum_{j=1}^{m} \left( \sum_{x_i \in Y_j \setminus Z_j} f(x_i) p_i \right) + \sum_{j=1}^{m} \left( \sum_{i=1}^{r_j} f(x_{j,i}) p_{j,i} \right)
\]

for \( f \in C(X) \). Note that the cardinality of \( \{x_i \in Y_j \setminus Z_j\} \) is \( KS_j \), \( j = 1, 2, \ldots, m \). Define

\[
\Psi(f) = \sum_{j=1}^{m} f(\xi_j) P_j = \sum_{k=1}^{K} \left( \sum_{j=1}^{m} f(\xi_j) Q_{j,k} \right) \text{ for all } f \in C(X),
\]

where \( P_j = \sum_{x_i \in Y_j \setminus Z_j} p_i = \sum_{k=1}^{K} Q_{j,k} \) and \( \text{rank} Q_{j,k} = S_j, j = 1, 2, \ldots, m \). Put \( e_0 = \sum_{i=1}^{m} (\sum_{i=1}^{r_j} p_{j,i}) \), \( e_k = \sum_{j=1}^{m} Q_{j,k}, k = 1, 2, \ldots, K \). Note that

\[
\text{rank}(e_0) = \sum_{j=1}^{m} r_j < mK \text{ and } \text{rank}(e_k) = S_j
\]

\[
S_j \geq N_0 mK > mK, \ j = 1, 2, \ldots, K.
\]

It follows that \( e_0 \lesssim e_1 \) and \( e_i \) is equivalent to \( e_1 \).
Moreover, we may write
\[
\Psi(f) = \text{diag}(\psi(f), \psi(f), \ldots, \psi(f)) \quad \text{for all } f \in A, \tag{e10.72}
\]
where \( \psi(f) = \sum_{j=1}^{m} f(\xi_j)Q_{j,1} \) for all \( f \in A \). We also estimate that
\[
\|H(f) - (\phi(f) \oplus \text{diag}(\psi(f), \psi(f), \ldots, \psi(f)))\| < \epsilon_1 \quad \text{for all } f \in F. \tag{e10.73}
\]
We also compute that
\[
\tau \circ \psi(g) \geq (1/K)(\Delta(\hat{g}) : g \in H_0) - \epsilon_1 - \frac{mK}{N_0Km} \geq \alpha \frac{\Delta(\hat{g})}{K} \quad \tag{e10.74}
\]
for all \( g \in H_0 \).
Cor (CorA) Let $A_0 = PM_r(C(X))P$, $A = A_0 \otimes C(\mathbb{T})$, let $\epsilon > 0$. 

Suppose that $H_1 \subset (A_0)^{1+\{0\}}$ is a finite subset, $\sigma > 0$ is a positive number and $n \geq 1$ is an integer. There exists a finite subset $H_2 \subset (A_0)^{1+\{0\}}$ satisfying the following: Suppose that $\phi: A = A_0 \otimes C(\mathbb{T}) \to M_k$ (for some integer $k \geq 1$) is a unital homomorphism and $tr \circ \phi(h \otimes 1) \geq \Delta(\hat{h})$ for all $h \in H_2$.

Then there exist mutually orthogonal projections $e_0, e_1, e_2, \ldots, e_n \in M_k$ such that $e_1, e_2, \ldots, e_n$ are equivalent and $\sum_{i=0}^{n} e_i = 1$, and there exists a unital homomorphism $\psi_0: A = A_0 \otimes C(\mathbb{T}) \to e_0 M_k e_0$ and $\psi: A = A_0 \otimes C(\mathbb{T}) \to e_1 M_k e_1$ such that one may write that $\|\phi(f) - \text{diag}(\psi_0(f), \psi(f), \ldots, \psi(f))\| < \epsilon$ (e 10.76) and $tr(e_0) < \sigma$ (e 10.77) for all $f \in F$, where $tr$ is the tracial state on $M_k$. 

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**Cor(CorA)** Let $A_0 = PM_r(C(X))P$, $A = A_0 \otimes C(\mathbb{T})$, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset.
Cor(CorA) Let $A_0 = PM_r(C(X))P$, $A = A_0 \otimes C(\mathbb{T})$, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. Let $\Delta : (A_0)^{q,1} \setminus \{0\} \to (0,1)$ be an order preserving map.

Suppose that $\mathcal{H}_1 \subset (A_0)^1 \setminus \{0\}$ is a finite subset, $\sigma > 0$ is positive number and $n \geq 1$ is an integer. There exists a finite subset $\mathcal{H}_2 \subset (A_0)^1 \setminus \{0\}$ satisfying the following: Suppose that $\phi : A = A_0 \otimes C(\mathbb{T}) \to M_k$ (for some integer $k \geq 1$) is a unital homomorphism and

$$tr \circ \phi(h \otimes 1) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_2.$$  \hspace{1cm} (e 10.75)

Then there exist mutually orthogonal projections $e_0, e_1, e_2, \ldots, e_n \in M_k$ such that $e_1, e_2, \ldots, e_n$ are equivalent and $\sum_{i=0}^n e_i = 1$, and there exists a unital homomorphisms $\psi_0 : A = A_0 \otimes C(\mathbb{T}) \to e_0 M_k e_0$ and $\psi : A = A_0 \otimes C(\mathbb{T}) \to e_1 M_k e_1$ such that one may write that

$$\|\phi(f) - \text{diag}(\psi_0(f), \psi(f), \psi(f), \ldots, \psi(f))\| < \epsilon \hspace{1cm} (e 10.76)$$

$$\text{and } tr(e_0) < \sigma \hspace{1cm} (e 10.77)$$

for all $f \in \mathcal{F}$, where $tr$ is the tracial state on $M_k$.  

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Moreover,

\[ tr(\psi(g \otimes 1)) \geq \frac{\Delta(\hat{g})}{2n} \quad \text{for all } g \in \mathcal{H}_1. \]  

(e10.78)
Lemma 2.6.

Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and let $A = PM_r(C(X))P$
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Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and let $A = PM_r(C(X))P$ and let $\Delta : A^q_{+1} \setminus \{0\} \to (0, 1)$ be an order preserving map.
Lemma 2.6.

Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and let $A = PM_r(C(X))P$ and let $\Delta : A_{+}^{q,1} \setminus \{0\} \to (0,1)$ be an order preserving map. For any $\epsilon > 0$ and any finite subset $F \subset A$, there exists $\delta > 0$, a finite subset $G \subset A$, a finite subset $P \subset K(A)$, a finite subset $H \subset A_{+}^{1} \setminus \{0\}$ and an integer $K \geq 1$ satisfying the following:

For any two unital $\delta$-G-multiplicative contractive completely positive linear maps $\phi_1, \phi_2 : A \to M_n$ (for some integer $n$) and any unital homomorphism $\psi : A \to M_m$ with $m \geq n$ such that $\tau \circ \psi(g) \geq \Delta(\hat{g})$ for all $g \in H$ and $[\phi_1]_P = [\phi_2]_P$, there exists a unitary $U \in M_{Km+n}$ such that $\|Ad_U \circ (\phi_1 \oplus \Psi)(f) - (\phi_2 \oplus \Psi)(f)\| < \epsilon$ for all $f \in A$, where $\Psi(f) = \text{diag}(K\psi(f), \psi(f), ..., \psi(f))$ for all $f \in A$. 

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Lemma 2.6.

Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and let $A = PM_r(C(X))P$ and let $\Delta : A_{+}^{q,1} \setminus \{0\} \to (0,1)$ be an order preserving map. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists $\delta > 0$, a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{P} \subset K(A)$,
Lemma 2.6.

Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and let $A = PM_r(C(X))P$ and let $\Delta : A^{q,1}_+ \setminus \{0\} \rightarrow (0, 1)$ be an order preserving map. For any $\epsilon > 0$ and any finite subset $F \subset A$, there exists $\delta > 0$, a finite subset $G \subset A$, a finite subset $P \subset K(A)$, a finite subset $H \subset A^{1}_+ \setminus \{0\}$ and an integer $K \geq 1$ satisfying the following:
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there exists a unitary $U \in M_{Kn+m}$ such that

$$\|\text{Ad } U \circ (\phi_1 \oplus \psi)(f) - (\phi_2 \oplus \psi)(f)\| < \epsilon \text{ for all } f \in A,$$ \hspace{1cm} (e10.80)
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$$\Psi(f) = \text{diag}(\psi(f), \psi(f), \ldots, \psi(f)) \text{ for all } f \in A.$$
The above follows from the following:

**Theorem 2.7.**

Let $A$ be a unital separable amenable $C^*$-algebra and let $B$ be a unital $C^*$-algebra.
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Let $A$ be a unital separable amenable $C^*$-algebra and let $B$ be a unital $C^*$-algebra. Suppose that $h_1, h_2 : A \to B$ are two homomorphisms such that

$$[h_1] = [h_2] \text{ in } KL(A, B).$$

Suppose that $h_0 : A \to B$ is a unital full monomorphism. Then, for any $\epsilon > 0$ and any finite subset $F \subset A$, there exists an integer $n \geq 1$ and a unitary $W \in M_{n+1}(B)$ such that

$$\|W^* \text{diag}(h_1(a), h_0(a), \ldots, h_0(a)) W - \text{diag}(h_2(a), h_0(a), \ldots, h_0(a))\| < \epsilon$$

for all $a \in F$ and $W^* p W = q$, where $p = \text{diag}(h_1(1_A), h_0(1_A), \ldots, h_0(1_A))$ and $q = \text{diag}(h_2(1_A), h_0(1_A), \ldots, h_0(1_A))$. In particular, if $h_1(1_A) = h_2(1_A)$, then $W \in U(pM_{n+1}(B)p)$. 

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Suppose that $h_1(1_A) = h_2(1_A)$, $W \in U(pM_{n+1}(B)p)$. 

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for all $a \in F$ and $W^* p W = q$, where $p = \text{diag}(h_1(1_A), h_0(1_A), ..., h_0(1_A))$ and $q = \text{diag}(h_2(1_A), h_0(1_A), ..., h_0(1_A))$. In particular, if $h_1(1_A) = h_2(1_A)$, $W \in U(p M_n + 1(B) p)$.
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Let $A$ be a unital separable amenable C*-algebra and let $B$ be a unital C*-algebra. Suppose that $h_1, h_2 : A \to B$ are two homomorphisms such that

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Suppose that $h_0 : A \to B$ is a unital full monomorphism. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists an integer $n \geq 1$ and a unitary $W \in M_{n+1}(B)$ such that
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for all $a \in \mathcal{F}$ and $W^* pW = q$, where

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In particular, if $h_1(1_A) = h_2(1_A)$, $W \in U(pM_{n+1}(B)p)$. 
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Let $A$ be a unital $C^*$-algebra whose irreducible representations have bounded dimension. Let $T$ be a finite subset of tracial states on $A$. For any finite subset $F \subset A$ and for any $\epsilon > 0$ and $\sigma > 0$, there is an ideal $J \subset A$ such that $\|\tau|_J\| < \sigma$ for all $\tau \in T$, a finite dimensional $C^*$-subalgebra $C \subset A/J$ and a unital homomorphism $\pi_0$ from $A$ such that $\text{dist}(\pi_0(a), C) < \epsilon$ for all $a \in F$,

\[ \pi_0(A) = \pi_0(C) \cong C \text{ and } \ker \pi_0 \supset J, \]

where $\pi : A \to A/J$ is the quotient map.
Lemma 2.8.

Let $A$ be a unital $C^*$-algebra whose irreducible representations have bounded dimension. Let $T$ be a finite subset of tracial states on $A$. For any finite subset $F \subset A$ and for any $\epsilon > 0$ and $\sigma > 0$, there is an ideal $J \subset A$ such that $\|\tau|_J\| < \sigma$ for all $\tau \in T$.\[\epsilon 10.81\]

\[\epsilon 10.82\]

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\[ \text{dist}(\pi_0(x), C) < \epsilon \quad \text{for all } x \in F, \]

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Proof: We prove the case that $A$ arising from a locally trivial continuous field of $M_n$ over a compact metric space $X$. We may assume that there is $\delta_0 > 0$ such that for any $x \in X$, $A|_{\bar{B}(x, \delta_0)} \cong M_n(C(\bar{B}(x, \delta_0)))$, where $\bar{B}(x, \delta_0) = \{y \in X: \text{dist}(x, y) \leq \delta_0\}$. Let $\epsilon > 0$. There exists $\delta_1 > 0$ such that $\|f(x) - f(x')\| < \epsilon/16$ for all $f \in F$, provided that $\text{dist}(x, x') < \delta_1$. We may assume that $\delta_1 < \delta_0$. For each $x \in X$, since $T$ is finite, there is $\delta_x$ with $\delta_1/2 < d_x < \delta_1$ such that $\mu_\tau(\{y: \text{dist}(x, y) = d_x\}) = 0$ for all $\tau \in T$, (10.83) where $\mu_\tau$ is the probability measure on $X$ induced by $\tau$. Note that $\bigcup_{x \in X} O(x, \delta_x) = X$. Suppose that $\bigcup_{i=1}^m O(x_i, \delta_{x_i}) = X$. Define $F = \bigcup_{i=1}^m \{y: \text{dist}(y, x_i) = \delta_{x_i}\}$. 

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Proof: We prove the case that $A$ arising from a locally trivial continuous field of $M_n$ over a compact metric space $X$. We may assume that there is $\delta_0 > 0$ such that for any $x \in X$, $A|_{\bar{B}(x,\delta_0)} \cong M_n(C(\bar{B}(x,\delta_0)))$, where $\bar{B}(x,\delta_0) = \{ y \in X : \text{dist}(x,y) \leq \delta_0 \}$. 

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For each $x \in X$, since $T$ is finite, there is $\delta_x$ with $\delta_1/2 < \delta_x < \delta_1$ such that $\mu_{\tau}(\{ y : \text{dist}(x,y) = \delta_x \}) = 0$ for all $\tau \in T$, (10.83) where $\mu_{\tau}$ is the probability measure on $X$ induced by $\tau$.

Note that $\bigcup_{x \in X} O(x,\delta_x) = X$. Suppose that $\bigcup_{m_i=1} O(x_i,\delta_{x_i}) = X$.

Define $F = \sum_{i=1}^m \{ y : \text{dist}(y,x_i) = \delta_{x_i} \}$. 

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provided that $\text{dist}(x, x') < \delta_1$. We may assume that $\delta_1 < \delta_0$.

For each $x \in X$, since $T$ is finite, there is $\delta_x$ with $\delta_1/2 < d_x < \delta_1$ such that

$$\mu_\tau(\{y : \text{dist}(x, y) = d_x\}) = 0 \text{ for all } \tau \in T,$$

(10.83)

where $\mu_\tau$ is the probability measure on $X$ induced by $\tau$. Note that $\bigcup_{x \in X} O(x, \delta_x) = X$. Suppose that $\bigcup_{i=1}^m O(x_i, \delta_{x_i}) = X$. 

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Proof: We prove the case that $A$ arising from a locally trivial continuous field of $M_n$ over a compact metric space $X$. We may assume that there is $\delta_0 > 0$ such that for any $x \in X$, $A|_{\bar{B}(x, \delta_0)} \cong M_n(C(\bar{B}(x, \delta_0)))$, where $\bar{B}(x, \delta_0) = \{ y \in X : \text{dist}(x, y) \leq \delta_0 \}$.

Let $\epsilon > 0$. There exists $\delta_1 > 0$ such that
\[ \| f(x) - f(x') \| < \epsilon/16 \text{ for all } f \in \mathcal{F}, \]
provided that $\text{dist}(x, x') < \delta_1$. We may assume that $\delta_1 < \delta_0$.

For each $x \in X$, since $T$ is finite, there is $\delta_x$ with $\delta_1/2 < d_x < \delta_1$ such that
\[ \mu_{\tau}(\{ y : \text{dist}(x, y) = d_x \}) = 0 \text{ for all } \tau \in T, \] (e10.83)
where $\mu_{\tau}$ is the probability measure on $X$ induced by $\tau$. Note that $\bigcup_{x \in X} O(x, \delta_x) = X$. Suppose that $\bigcup_{i=1}^m O(x_i, \delta_{x_i}) = X$. Define
\[ F = \sum_{i=1}^m \{ y : \text{dist}(y, x_i) = \delta_{x_i} \}. \]
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By the choice of $\delta_1$, we estimate that, for any $f \in F$,

$$\text{dist} (f|_{G_i}, C_i) < \frac{\epsilon}{16}.$$ 

It follows that $\text{dist}(\pi(f), C) < \epsilon$ for all $f \in F$.

Let $\pi_0 : A \to \bigoplus_{i=1}^K M_n$ be defined by $\pi_0(f) = \bigoplus_{i=1}^K f(\xi_i)$ for all $f \in A$. Then $\ker \pi_0 \supset J$ and $\pi_0(A) = \pi_0(C) \sim C$. 

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Theorem 2.9. (L. G. Brown–1991) Let $A$ be a $C^*$-algebra, $p$ and $q$ are two closed projections (in $A^{**}$) such that $pq = 0$ and $p$ is compact. Then there exists a projection $e \in A$ such that $p \leq e \leq (1 - q)$. The converse also holds.
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Let

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Suppose also that every projection $\bar{e} \in C$ can be lifted, i.e., there is a projection $e \in A$ such that $\pi(e) = \bar{e}$, where $\pi : A \to C$ is the quotient map.

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**Proof:** For any $r > 0$, define $f \in C_0((0, \infty))_+$ as follows.
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(Note 0 ≤ $h_n(a) ≤ h_{n+1}(a)$, $n = 1, 2, ...$)

Let $h = \lim_{n \to \infty} h_n(a)$ in $A^{\ast \ast}$ which corresponds to the open interval $(-\infty, \frac{1}{8})$. $h$ is an open projection. Let $p = 1 - h$ (or $\chi_{[1/8, 1]}(a)$). $p$ is compact. Let $1 - q$ be the open projection corresponds to $(1/16, \infty)$, i.e., $1 - q = \lim_{n \to \infty} (f_{1/16}(a))^1/n$. So $q$ is closed and $pq = 0$. 

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It follows Brown interpolation lemma that there is a projection \( e \in D \) such that

\[
p \leq e \leq 1 - q. \tag{e10.84}
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We have

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Lemma 2.12.

Let $A$ be a unital $C^*$-algebra whose irreducible representations have bounded dimensions.
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Suppose that $\phi : A \to M_n$ (for some integer $n \geq 1$) is a $\delta$-G-multiplicative contractive completely positive linear map. Then, there exists a projection $p \in M_n$ and a unital homomorphism $\phi_0 : A \to pM_n p$ such that

\[
\|p \phi(a) - \phi(a)p\| < \epsilon
\]

for all $a \in \mathcal{F}$,

\[
\|\phi(a) - [(1 - p) \phi(a)(1 - p) + \phi_0(a)]\| < \epsilon
\]

for all $a \in \mathcal{F}$ and

\[
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Huaxin Lin 
Basic Homotopy Lemmas Introduction
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\|\phi(a) - [(1 - p)\phi(a)(1 - p) + \phi_0(a)]\| < \epsilon \quad \text{for all } a \in \mathcal{F} \quad \text{and}
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Let $A$ be a unital $C^*$-algebra whose irreducible representations have bounded dimensions. Let $\epsilon > 0$, let $\mathcal{F} \subset A$ be a finite subset and let $\sigma_0 > 0$. There exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: Suppose that $\phi : A \to M_n$ (for some integer $n \geq 1$) is a $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear map. Then, there exists a projection $p \in M_n$ and a unital homomorphism $\phi_0 : A \to pM_n p$ such that

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Lemma 2.12.
Let $A$ be a unital $C^*$-algebra whose irreducible representations have bounded dimensions. Let $\epsilon > 0$, let $F \subset A$ be a finite subset and let $\sigma_0 > 0$. There exist $\delta > 0$ and a finite subset $G \subset A$ satisfying the following: Suppose that $\phi : A \to M_n$ (for some integer $n \geq 1$) is a $\delta$-$G$-multiplicative contractive completely positive linear map. Then, there exists a projection $p \in M_n$ and a unital homomorphism $\phi_0 : A \to pM_np$ such that

$$\|p\phi(a) - \phi(a)p\| < \epsilon \text{ for all } a \in F,$$
$$\|\phi(a) - [(1 - p)\phi(a)(1 - p) + \phi_0(a)]\| < \epsilon \text{ for all } a \in F \text{ and }$$
$$\text{tr}(1 - p) < \sigma_0,$$

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**Proof**: We assume that the lemma is false.
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Denote by $\bigoplus_{n=1}^{\infty}(\{M_{m(n)}\})$ the ideal
\[
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Denote by $Q$ the quotient $\prod_{n=1}^{\infty}(\{M_{m(n)}\}) / \bigoplus_{n=1}^{\infty}(\{M_{m(n)}\})$. Let $\pi_\omega : \prod_{n=1}^{\infty}(\{M_{m(n)}\}) \to Q$ be the quotient map. Let $A_0 = \{\pi_\omega(\{\phi_n(f)\}) : f \in A\}$ which is a subalgebra of $Q$. 

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It follows from Lemma 2.8 that there is a (two-sided closed) ideal $I \subset \Psi(A)$.
Denote by \( \bigoplus_{n=1}^{\infty} \{ M_{m(n)} \} \) the ideal

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Let \( A_0 = \{ \pi_\omega(\{ \phi_n(f) \}) : f \in A \} \) which is a subalgebra of \( Q \). Then \( \Psi \) is a unital homomorphism from \( A \) to \( \prod_{n=1}^{\infty} (M_{m(n)}) / \bigoplus (\{ M_{m(n)} \}) \) with \( \Psi(A) = \pi_\omega(A_0) \). If \( a \in A \) has zero image in \( \pi_\omega(A_0) \), that is, \( \phi_n(a) \to 0 \), then \( t_0(a) = \lim_{n \to \infty} tr_n(\phi_n(a)) = 0 \). So we may view \( t_0 \) as a state on \( \pi_\omega(A_0) = \Psi(A) \).
It follows from Lemma 2.8 that there is a (two-sided closed) ideal \( I \subset \Psi(A) \) and a finite dimensional \( C^* \)-subalgebra \( B \subset \Psi(A)/I \) and a unital homomorphism \( \pi_{00} : \Psi(A)/I \to B \) such that
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$$\text{dist}(\pi_I \circ \Psi(f), B) < \epsilon_0/16 \text{ for all } f \in F_0, \quad (e10.87)$$

$$\|(t_0)_I\| < \sigma_0/2 \quad (e10.88)$$

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\bigoplus_{n=1}^{\infty}(\{M_{m(n)}\}) = \{\{a_n\} : a_n \in M_{m(n)} \text{ and } \lim_{n \to \infty} \|a_n\| = 0\}.
\]

Denote by $Q$ the quotient $\prod_{n=1}^{\infty}(\{M_{m(n)}\}) / \bigoplus_{n=1}^{\infty}(\{M_{m(n)}\})$. Let $\pi_\omega : \prod_{n=1}^{\infty}(\{M_{m(n)}\}) \to Q$ be the quotient map.

Let $A_0 = \{\pi_\omega(\{\phi_n(f)\}) : f \in A\}$ which is a subalgebra of $Q$. Then $\Psi$ is a unital homomorphism from $A$ to $\prod_{n=1}^{\infty}(M_{m(n)}) / \bigoplus(\{M_{m(n)}\})$ with $\Psi(A) = \pi_\omega(A_0)$. If $a \in A$ has zero image in $\pi_\omega(A_0)$, that is, $\phi_n(a) \to 0$, then $t_0(a) = \lim_{n \to \infty} tr_n(\phi_n(a)) = 0$. So we may view $t_0$ as a state on $\pi_\omega(A_0) = \Psi(A)$.

It follows from Lemma 2.8 that there is a (two-sided closed) ideal $I \subset \Psi(A)$ and a finite dimensional $C^*$-subalgebra $B \subset \Psi(A)/I$ and a unital homomorphism $\pi_{00} : \Psi(A)/I \to B$ such that
\[
\dist(\pi_I \circ \Psi(f), B) < \frac{\epsilon_0}{16} \text{ for all } f \in \mathcal{F}_0, \quad (e\,10.87)
\]
\[
\|t_0\|_I < \frac{\sigma_0}{2} \quad (e\,10.88)
\]
\[
\pi_{00}|_B = \text{id}. \quad (e\,10.89)
\]

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$$\|\pi I \circ \Psi(f) - b_f\| < \epsilon_0 / 16.$$ (e 10.90)

Put $C' = B + I$ and $I_0 = \Psi^{-1}(I)$ and $C_1 = \Psi^{-1}(C')$. For each $f \in F_0$, there exists $a_f \in C_1 \subset A$ such that

$$\|f - a_f\| < \epsilon_0 / 16$$ and $\pi I \circ \Psi(a_f) = b_f$. (e 10.91)

Let $a \in (I_0) + I$ be a strictly positive element and let $J = \Psi(a)\Psi(a)$ be the hereditary $C^*$-subalgebra of $Q$ generated by $\Psi(a)$. Put $C_2 = \Psi(C_1) + J$. Then $J$ is an ideal of $C_2$. Denote by $\pi_J: C_2 \to B$ the quotient map. Since $Q$ and $J$ have real rank zero and $C_2/J$ has finite dimensional, $C_2$ has real rank zero. It follows that $0 \to J \to C_2 \to B \to 0$ is a quasidiagonal extension.

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is a quasidiagonal extension.
One then concludes there is a projection $P \in J$ and a unital homomorphism $\psi_0 : B \to (1 - P)C_2(1 - P)$ such that

$$\|P\psi(a_f) - \psi(a_f)P\| < \epsilon_0/8 \quad \text{and} \quad (e10.92)$$
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$$\|P\psi(f) - \psi(f)P\| < \epsilon_0/2 \quad \text{and}$$

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Note that $\dim H(A) < \infty$, and that $H(A) \subset Q$. There is a homomorphism $H_1 : H(A) \to \prod_{n=1}^{\infty} \left\{ M_{m}^{(n)} \right\}$ such that $\pi \circ H_1 \circ H = H$.

One may write $H_1 = \{ h_n \}$, where each $h_n : H(A) \to M_{m}^{(n)}$ is a (not necessary unital) homomorphism, $n = 1, 2, ...$. There is also a sequence of projections $q_n \in M_{m}^{(n)}$ such that $\pi(\{ q_n \}) = P$.
One then concludes there is a projection $P \in J$ and a unital homomorphism $\psi_0 : B \to (1 - P)C_2(1 - P)$ such that

\begin{align*}
\|P\psi(a_f) - \psi(a_f)P\| &< \epsilon_0/8 \quad \text{and} \\
\|\psi(a_f) - [P\psi(a_f)P + \psi_0 \circ \pi_J \circ \psi(a_f)]\| &< \epsilon_0/8 
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Then, for sufficiently large $n$, by (e10.94) and (e10.95),

$$\|(1 - p_n)\phi_n(f) - \phi_n(f)(1 - p_n)\| < \epsilon_0,$$  

(e10.96)
Then, for sufficiently large $n$, by (e 10.94) and (e 10.95),

\[
\| (1 - p_n) \phi_n(f) - \phi_n(f)(1 - p_n) \| < \epsilon_0, \tag{e 10.96}
\]

\[
\| \phi_n(f) - [(1 - p_n) \phi_n(f)(1 - p_n) + h_n \circ H(f)] \| < \epsilon_0 \tag{e 10.97}
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\[ \| \Psi(b)P - P \| < \eta. \]
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\| \Psi(b)P - P \| < \eta.
\]

However, by (e 10.88),

\[
0 < t_0(\Psi(b)) < \sigma_0/2 \text{ for all } b \in l_0 \text{ with } 0 \leq n \leq 1. \tag{e 10.98}
\]
Then, for sufficiently large $n$, by (e 10.94) and (e 10.95),

$$\|(1 - p_n)\phi_n(f) - \phi_n(f)(1 - p_n)\| < \epsilon_0,$$  \hspace{1cm} \text{(e 10.96)}

$$\|\phi_n(f) - [(1 - p_n)\phi_n(f)(1 - p_n) + h_n \circ H(f)]\| < \epsilon_0 \quad \text{(e 10.97)}$$

for all $f \in \mathcal{F}_0$. Moreover, since $P \in J$, for any $\eta > 0$, there is $b \in l_0$ with $0 \leq b \leq 1$ such that

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By choosing sufficiently small $\eta$, for all sufficiently large $n$, 

Then, for sufficiently large \( n \), by (e 10.94) and (e 10.95),

\[
\| (1 - p_n) \phi_n(f) - \phi_n(f)(1 - p_n) \| < \epsilon_0, \tag{e 10.96}
\]

\[
\| \phi_n(f) - [(1 - p_n) \phi_n(f)(1 - p_n) + h_n \circ H(f)] \| < \epsilon_0 \tag{e 10.97}
\]

for all \( f \in \mathcal{F}_0 \). Moreover, since \( P \in J \), for any \( \eta > 0 \), there is \( b \in I \) with \( 0 \leq b \leq 1 \) such that

\[
\| \Psi(b)P - P \| < \eta.
\]

However, by (e 10.88),

\[
0 < t_0(\Psi(b)) < \sigma_0/2 \text{ for all } b \in I \text{ with } 0 \leq n \leq 1. \tag{e 10.98}
\]

By choosing sufficiently small \( \eta \), for all sufficiently large \( n \),

\[
tr_n(1 - p_n) < \sigma_0.
\]
Then, for sufficiently large $n$, by (e10.94) and (e10.95),

\[ \| (1 - p_n)\phi_n(f) - \phi_n(f)(1 - p_n) \| < \varepsilon_0, \]  
\[ \| \phi_n(f) - [(1 - p_n)\phi_n(f)(1 - p_n) + h_n \circ H(f)] \| < \varepsilon_0 \]  

for all $f \in \mathcal{F}_0$. Moreover, since $P \in J$, for any $\eta > 0$, there is $b \in I_0$ with $0 \leq b \leq 1$ such that

\[ \| \Psi(b)P - P \| < \eta. \]

However, by (e10.88),

\[ 0 < t_0(\Psi(b)) < \sigma_0/2 \]  
for all $b \in I_0$ with $0 \leq n \leq 1$.  

By choosing sufficiently small $\eta$, for all sufficiently large $n$,  

\[ tr_n(1 - p_n) < \sigma_0. \]

This contradicts with (e10.86).
Corollary 2.13.

Let $A$ be a unital $C^*$-algebra whose irreducible representations have bounded dimensions.
Corollary 2.13.

Let $A$ be a unital $C^*$-algebra whose irreducible representations have bounded dimensions. Let $\eta > 0$, let $\mathcal{E} \subset A$ be a finite subset and let $\eta_0 > 0$. There exist $\delta > 0$ and a finite subset $G \subset A$ satisfying the following:

Suppose that $\phi, \psi : A \to M_n$ (for some integer $n \geq 1$) are two $G$-$\delta$-multiplicative contractive completely positive linear maps. Then, there exist projections $p, q \in M_n$ with $\operatorname{rank}(p) = \operatorname{rank}(q)$ and unital homomorphisms $\phi_0 : A \to pM_n p$ and $\psi_0 : A \to qM_n q$ such that

\[
\|p \phi(a) - \phi(a) p\| < \eta,
\]

\[
\|q \psi(a) - \psi(a) q\| < \eta,
\]

$a \in \mathcal{E}$, and

\[
\operatorname{tr}(1 - p) = \operatorname{tr}(1 - q) < \eta_0.
\]

Where $\operatorname{tr}$ is the normalized trace on $M_n$. 
Corollary 2.13.

Let $A$ be a unital $C^*$-algebra whose irreducible representations have bounded dimensions. Let $\eta > 0$, let $E \subset A$ be a finite subset and let $\eta_0 > 0$. There exist $\delta > 0$ and a finite subset $G \subset A$ satisfying the following:
**Corollary 2.13.**

Let $A$ be a unital $C^*$-algebra whose irreducible representations have bounded dimensions. Let $\eta > 0$, let $E \subset A$ be a finite subset and let $\eta_0 > 0$. There exist $\delta > 0$ and a finite subset $G \subset A$ satisfying the following: Suppose that $\phi, \psi : A \to M_n$ (for some integer $n \geq 1$) are two $G$-$\delta$-multiplicative contractive completely positive linear maps. There exist projections $p, q \in M_n$ with $\text{rank}(p) = \text{rank}(q)$ and unital homomorphisms $\phi_0 : A \to pM_n p$ and $\psi_0 : A \to qM_n q$ such that

\[
\|p\phi(a) - \phi(a)p\| < \eta, \quad \|q\psi(a) - \psi(a)q\| < \eta,
\]

for all $a \in E$, and

\[
\text{tr}(1 - p) = \text{tr}(1 - q) < \eta_0,
\]

where tr is the normalized trace on $M_n$. 

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Corollary 2.13.

Let $A$ be a unital $C^*$-algebra whose irreducible representations have bounded dimensions. Let $\eta > 0$, let $E \subset A$ be a finite subset and let $\eta_0 > 0$. There exist $\delta > 0$ and a finite subset $G \subset A$ satisfying the following: Suppose that $\phi, \psi : A \rightarrow M_n$ (for some integer $n \geq 1$) are two $G$-$\delta$-multiplicative contractive completely positive linear maps. Then, there exist projections $p, q \in M_n$ with $\text{rank}(p) = \text{rank}(q)$
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Corollary 2.13.

Let $A$ be a unital $C^*$-algebra whose irreducible representations have bounded dimensions. Let $\eta > 0$, let $\mathcal{E} \subset A$ be a finite subset and let $\eta_0 > 0$. There exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: Suppose that $\phi, \psi : A \to M_n$ (for some integer $n \geq 1$) are two $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps. Then, there exist projections $p, q \in M_n$ with $\text{rank}(p) = \text{rank}(q)$ and unital homomorphisms $\phi_0 : A \to pM_n p$ and $\psi_0 : A \to qM_n q$ such that

\[
\|p\phi(a) - \phi(a)p\| < \eta, \quad \|q\psi(a) - \psi(a)q\| < \eta, \quad a \in \mathcal{E},
\]
**Corollary 2.13.**

Let $A$ be a unital C$^*$-algebra whose irreducible representations have bounded dimensions. Let $\eta > 0$, let $\mathcal{E} \subset A$ be a finite subset and let $\eta_0 > 0$. There exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: Suppose that $\phi, \psi : A \to M_n$ (for some integer $n \geq 1$) are two $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps. Then, there exist projections $p, q \in M_n$ with $\operatorname{rank}(p) = \operatorname{rank}(q)$ and unital homomorphisms $\phi_0 : A \to pM_n p$ and $\psi_0 : A \to qM_n q$ such that

\[
\| p \phi(a) - \phi(a)p \| < \eta, \quad \| q \psi(a) - \psi(a)q \| < \eta, \quad a \in \mathcal{E},
\]

\[
\| \phi(a) - [(1 - p)\phi(a)(1 - p) + \phi_0(a)] \| < \eta,
\]
**Corollary 2.13.**

Let $A$ be a unital $C^*$-algebra whose irreducible representations have bounded dimensions. Let $\eta > 0$, let $\mathcal{E} \subset A$ be a finite subset and let $\eta_0 > 0$. There exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: Suppose that $\phi, \psi : A \to M_n$ (for some integer $n \geq 1$) are two $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps. Then, there exist projections $p, q \in M_n$ with $\text{rank}(p) = \text{rank}(q)$ and unital homomorphisms $\phi_0 : A \to pM_n p$ and $\psi_0 : A \to qM_n q$ such that

\[
\|p \phi(a) - \phi(a)p\| < \eta, \quad \|q \psi(a) - \psi(a)q\| < \eta, \quad a \in \mathcal{E},
\]

\[
\|\phi(a) - [(1 - p)\phi(a)(1 - p) + \phi_0(a)]\| < \eta,
\]

\[
\|\psi(a) - [(1 - q)\psi(a)(1 - q) + \psi_0(a)]\| < \eta, \quad a \in \mathcal{E}
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Let $A$ be a unital $C^*$-algebra whose irreducible representations have bounded dimensions. Let $\eta > 0$, let $\mathcal{E} \subset A$ be a finite subset and let $\eta_0 > 0$. There exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: Suppose that $\phi, \psi : A \to M_n$ (for some integer $n \geq 1$) are two $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps. Then, there exist projections $p, q \in M_n$ with $\text{rank}(p) = \text{rank}(q)$ and unital homomorphisms $\phi_0 : A \to pM_np$ and $\psi_0 : A \to qM_nq$ such that

$$
\| p\phi(a) - \phi(a)p \| < \eta, \quad \| q\psi(a) - \psi(a)q \| < \eta, \quad a \in \mathcal{E},
$$
$$
\| \phi(a) - [(1 - p)\phi(a)(1 - p) + \phi_0(a)] \| < \eta,
$$
$$
\| \psi(a) - [(1 - q)\psi(a)(1 - q) + \psi_0(a)] \| < \eta, \quad a \in \mathcal{E}
$$
and $\text{tr}(1 - p) = \text{tr}(1 - q) < \eta_0$, where $\text{tr}$ is the normalized trace on $M_n$.  

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Corollary 2.13.

Let $A$ be a unital $C^*$-algebra whose irreducible representations have bounded dimensions. Let $\eta > 0$, let $\mathcal{E} \subset A$ be a finite subset and let $\eta_0 > 0$. There exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: Suppose that $\phi, \psi : A \to M_n$ (for some integer $n \geq 1$) are two $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps. Then, there exist projections $p, q \in M_n$ with $\text{rank}(p) = \text{rank}(q)$ and unital homomorphisms $\phi_0 : A \to pM_n p$ and $\psi_0 : A \to qM_n q$ such that

\[
\|p \phi(a) - \phi(a)p\| < \eta, \quad \|q \psi(a) - \psi(a)q\| < \eta, \quad a \in \mathcal{E},
\]

\[
\|\phi(a) - [(1 - p)\phi(a)(1 - p) + \phi_0(a)]\| < \eta,
\]

\[
\|\psi(a) - [(1 - q)\psi(a)(1 - q) + \psi_0(a)]\| < \eta, \quad a \in \mathcal{E}
\]

and $\text{tr}(1 - p) = \text{tr}(1 - q) < \eta_0,$

where $\text{tr}$ is the normalized trace on $M_n$. 
Lemma 2.14.
Let $A$ be an infinite dimensional unital sub-homogeneous $C^*$-algebra,
Lemma 2.14.

Let $A$ be an infinite dimensional unital sub-homogeneous $C^*$-algebra, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset.
Lemma 2.14.

Let $A$ be an infinite dimensional unital sub-homogeneous $C^*$-algebra, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. Let $\epsilon_0 > 0$ and let $\mathcal{G}_0 \subset A$ be a finite subset.
Lemma 2.14.
Let $A$ be an infinite dimensional unital sub-homogeneous $C^*$-algebra, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. Let $\epsilon_0 > 0$ and let $\mathcal{G}_0 \subset A$ be a finite subset. Let $\Delta : A^{q,1}_+ \setminus \{0\} \to (0,1)$ be a positive map.
Lemma 2.14.

Let $A$ be an infinite dimensional unital sub-homogeneous $C^*$-algebra, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. Let $\epsilon_0 > 0$ and let $\mathcal{G}_0 \subset A$ be a finite subset. Let $\Delta : A^q_{++} \setminus \{0\} \to (0,1)$ be a positive map. Suppose that $\mathcal{H}_1 \subset A^1_{++} \setminus \{0\}$ is a finite subset,
Lemma 2.14.

Let $A$ be an infinite dimensional unital sub-homogeneous $C^*$-algebra, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. Let $\epsilon_0 > 0$ and let $\mathcal{G}_0 \subset A$ be a finite subset. Let $\Delta : A^{q,1}_+ \setminus \{0\} \to (0,1)$ be a positive map. Suppose that $\mathcal{H}_1 \subset A^{1}_+ \setminus \{0\}$ is a finite subset, $\epsilon_1 > 0$ is a positive number and $K \geq 1$ is an integer.
Lemma 2.14.

Let $A$ be an infinite dimensional unital sub-homogeneous $C^*$-algebra, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. Let $\epsilon_0 > 0$ and let $\mathcal{G}_0 \subset A$ be a finite subset. Let $\Delta : A^q_+ \setminus \{0\} \to (0,1)$ be a positive map. Suppose that $\mathcal{H}_1 \subset A^1_+ \setminus \{0\}$ is a finite subset, $\epsilon_1 > 0$ is a positive number and $K \geq 1$ is an integer. There exists $\delta > 0$, $\sigma > 0$ and a finite subset $\mathcal{G} \subset A$. 
Lemma 2.14.

Let $A$ be an infinite dimensional unital sub-homogeneous $C^*$-algebra, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. Let $\epsilon_0 > 0$ and let $\mathcal{G}_0 \subset A$ be a finite subset. Let $\Delta : A_+^{q,1} \setminus \{0\} \to (0,1)$ be a positive map. Suppose that $\mathcal{H}_1 \subset A_+^{1} \setminus \{0\}$ is a finite subset, $\epsilon_1 > 0$ is a positive number and $K \geq 1$ is an integer. There exists $\delta > 0$, $\sigma > 0$ and a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{H}_2 \subset A_+^{1} \setminus \{0\}$ satisfying the following:
Lemma 2.14.

Let $A$ be an infinite dimensional unital sub-homogeneous $C^*$-algebra, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. Let $\epsilon_0 > 0$ and let $\mathcal{G}_0 \subset A$ be a finite subset. Let $\Delta : A_+^{q,1} \setminus \{0\} \to (0,1)$ be a positive map. Suppose that $\mathcal{H}_1 \subset A_+^1 \setminus \{0\}$ is a finite subset, $\epsilon_1 > 0$ is a positive number and $K \geq 1$ is an integer. There exists $\delta > 0$, $\sigma > 0$ and a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{H}_2 \subset A_+^1 \setminus \{0\}$ satisfying the following: Suppose that $L_1, L_2 : A \to M_n$ (for some integer $n \geq 1$) are unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear maps.
Lemma 2.14.

Let $A$ be an infinite dimensional unital sub-homogeneous $C^*$-algebra, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. Let $\epsilon_0 > 0$ and let $\mathcal{G}_0 \subset A$ be a finite subset. Let $\Delta : A^q_+ \setminus \{0\} \to (0, 1)$ be a positive map. Suppose that $\mathcal{H}_1 \subset A^1_+ \setminus \{0\}$ is a finite subset, $\epsilon_1 > 0$ is a positive number and $K \geq 1$ is an integer. There exists $\delta > 0$, $\sigma > 0$ and a finite subset $\mathcal{G} \subset A$ and a finite subset $\mathcal{H}_2 \subset A^1_+ \setminus \{0\}$ satisfying the following:

Suppose that $L_1, L_2 : A \to M_n$ (for some integer $n \geq 1$) are unital $\delta$-$\mathcal{G}$-multiplicative contractive completely positive linear maps

$$tr \circ L_1(h) \geq \Delta(\hat{h}) \text{ and } tr \circ L_2(h) \geq \Delta(\hat{h}) \quad \text{for all } h \in \mathcal{H}_2,$$

and

$$|tr \circ L_1(h) - tr \circ L_2(h)| < \sigma \quad \text{for all } h \in \mathcal{H}_2. \quad (e10.99)$$
Then there exist mutually orthogonal projections $e_0, e_1, e_2, \ldots, e_K \in M_n$ such that $e_1, e_2, \ldots, e_K$ are equivalent, $e_0 \lesssim e_1$, $tr(e_0) < \epsilon_1$ and $e_0 + \sum_{i=1}^K e_i = 1$, and there exist a unital $\epsilon_0$-$G_0$-multiplicative contractive completely positive linear maps $\psi_1, \psi_2 : A \to e_0 M_k e_0$, a unital homomorphism $\psi : A \to e_1 M_k e_1$, and unitary $u \in M_n$ such that one may write that

$$\left\| L_1(f) - \text{diag}(\psi_1(f), \psi(f), \psi(f), \ldots, \psi(f)) \right\| < \epsilon \quad \text{and} \quad (e10.100)$$

$$\left\| u L_2(f) u^* - \text{diag}(\psi_2(f), \psi(f), \psi(f), \ldots, \psi(f)) \right\| < \epsilon \quad (e10.101)$$

for all $f \in \mathcal{F}$, where $tr$ is the tracial state on $M_n$. Moreover,

$$tr(\psi(g)) \geq \frac{\Delta(\hat{g})}{3K} \quad \text{for all} \ g \in \mathcal{H}_1. \quad (e10.102)$$
Theorem 2.1. Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. 
Theorem 2.1. Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. Let $\Delta : C^{q,1}_+ \setminus \{0\} \to (0,1)$ be an order preserving map.
Theorem 2.1. Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. Let $\Delta : C^{q,1}_+ \setminus \{0\} \to (0,1)$ be an order preserving map. Let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset.
Theorem 2.1. Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. Let $\Delta : C^q_{+} \setminus \{0\} \to (0,1)$ be an order preserving map. Let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. There exists a finite subset $\mathcal{H}_1 \subset A_+ \setminus \{0\}$,
Theorem 2.1. Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. Let $\Delta : C^q_+ \setminus \{0\} \to (0,1)$ be an order preserving map. Let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. There exists a finite subset $\mathcal{H}_1 \subset A_+ \setminus \{0\}$, a finite subset $\mathcal{G} \subset A$, and $\sigma > 0$ satisfying the following: Suppose that $L_1, L_2 : A \to M_k$ (for some integer $k \geq 1$) are two unital $\mathcal{G}$-contractive completely positive linear maps such that $\|L_1|_P - L_2|_P\|$, $\text{tr} \circ L_1(h) \geq \Delta(\hat{h})$, $\text{tr} \circ L_2(h) \geq \Delta(\hat{h})$ for all $h \in \mathcal{H}_1$ and $\|\text{tr} \circ L_1(h) - \text{tr} \circ L_2(h)\| < \sigma$ for all $h \in \mathcal{H}_2$, then there exists a unitary $u \in M_k$ such that $\|\text{Ad}_u \circ L_1(f) - L_2(f)\| < \epsilon$ for all $f \in \mathcal{F}$. (e 10.103)
Theorem 2.1. Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. Let $\Delta : C^1_+ \setminus \{0\} \to (0,1)$ be an order preserving map. Let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. There exists a finite subset $\mathcal{H}_1 \subset A_+ \setminus \{0\}$, a finite subset $\mathcal{G} \subset A$, $\delta > 0$, a finite subset $\mathcal{P} \subset K(A)$, \[ \|\text{Ad}u \circ L_1(f) - L_2(f)\| < \epsilon \] for all $f \in \mathcal{F}$.
Theorem 2.1. Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. Let $\Delta : C^*_+ \setminus \{0\} \to (0,1)$ be an order preserving map. Let $\epsilon > 0$ and let $F \subset A$ be a finite subset.

There exists a finite subset $\mathcal{H}_1 \subset A_+ \setminus \{0\}$, a finite subset $\mathcal{G} \subset A$, $\delta > 0$, a finite subset $\mathcal{P} \subset \mathcal{K}(A)$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ and $\sigma > 0$ satisfying the following:
Theorem 2.1. Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. Let $\Delta : C^1_+ \setminus \{0\} \to (0, 1)$ be an order preserving map. Let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. There exists a finite subset $\mathcal{H}_1 \subset A_+ \setminus \{0\}$, a finite subset $\mathcal{G} \subset A$, $\delta > 0$, a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ and $\sigma > 0$ satisfying the following: Suppose that $L_1, L_2 : A \to M_k$ (for some integer $k \geq 1$) are two unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps
Theorem 2.1. Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. Let $\Delta : C_+^{q,1} \setminus \{0\} \to (0,1)$ be an order preserving map. Let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. There exists a finite subset $\mathcal{H}_1 \subset A_+ \setminus \{0\}$, a finite subset $\mathcal{G} \subset A$, $\delta > 0$, a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ and $\sigma > 0$ satisfying the following: Suppose that $L_1, L_2 : A \to M_k$ (for some integer $k \geq 1$) are two unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps such that

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}},$$
Theorem 2.1. Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. Let $\Delta : C^q_{+} \setminus \{0\} \rightarrow (0, 1)$ be an order preserving map. Let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. There exists a finite subset $\mathcal{H}_1 \subset A_+ \setminus \{0\}$, a finite subset $\mathcal{G} \subset A$, $\delta > 0$, a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ and $\sigma > 0$ satisfying the following: Suppose that $L_1, L_2 : A \rightarrow M_k$ (for some integer $k \geq 1$) are two unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps such that

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}},$$

$$\text{tr} \circ L_1(h) \geq \Delta(\hat{h}), \quad \text{tr} \circ L_2(h) \geq \Delta(\hat{h}) \quad \text{for all} \ h \in \mathcal{H}_1$$
Theorem 2.1. Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. Let $\Delta : C^1_+ \setminus \{0\} \to (0, 1)$ be an order preserving map. Let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. There exists a finite subset $\mathcal{H}_1 \subset A_+ \setminus \{0\}$, a finite subset $\mathcal{G} \subset A$, $\delta > 0$, a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A_s.a.$ and $\sigma > 0$ satisfying the following: Suppose that $L_1, L_2 : A \to M_k$ (for some integer $k \geq 1$) are two unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps such that

\[
[L_1]_{\mathcal{P}} = [L_2]_{\mathcal{P}},
\]

\[
\text{tr} \circ L_1(h) \geq \Delta(\hat{h}), \quad \text{tr} \circ L_2(h) \geq \Delta(\hat{h}) \quad \text{for all } h \in \mathcal{H}_1
\]

and

\[
|\text{tr} \circ L_1(h) - \text{tr} \circ L_2(h)| < \sigma \quad \text{for all } h \in \mathcal{H}_2,
\]

then there exists a unitary $u \in M_k$ such that

\[
\|\text{Ad}_u \circ L_1(f) - L_2(f)\| < \epsilon
\]
Theorem 2.1. Let \( X \) be a compact metric space, \( P \in M_r(C(X)) \) be a projection and \( C = PM_r(C(X)) \). Let \( \Delta : C^{q,1}_+ \backslash \{0\} \to (0,1) \) be an order preserving map. Let \( \epsilon > 0 \) and let \( \mathcal{F} \subset A \) be a finite subset. There exists a finite subset \( \mathcal{H}_1 \subset A_+ \backslash \{0\} \), a finite subset \( \mathcal{G} \subset A \), \( \delta > 0 \), a finite subset \( \mathcal{P} \subset K(A) \), a finite subset \( \mathcal{H}_2 \subset A_{s.a.} \) and \( \sigma > 0 \) satisfying the following: Suppose that \( L_1, L_2 : A \to M_k \) (for some integer \( k \geq 1 \)) are two unital \( \mathcal{G} \)-\( \delta \)-multiplicative contractive completely positive linear maps such that

\[
[L_1]|_\mathcal{P} = [L_2]|_\mathcal{P}, \\
tr \circ L_1(h) \geq \Delta(\hat{h}), \quad tr \circ L_2(h) \geq \Delta(\hat{h}) \quad \text{for all } h \in \mathcal{H}_1 \\
\text{and} \quad |tr \circ L_1(h) - tr \circ L_2(h)| < \sigma \quad \text{for all } h \in \mathcal{H}_2,
\]

then there exists a unitary \( u \in M_k \) such that
**Theorem 2.1.** Let $X$ be a compact metric space, $P \in M_r(C(X))$ be a projection and $C = PM_r(C(X))$. Let $\Delta : C_+^{q,1} \setminus \{0\} \to (0,1)$ be an order preserving map. Let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. There exists a finite subset $\mathcal{H}_1 \subset A_+ \setminus \{0\}$, a finite subset $\mathcal{G} \subset A$, $\delta > 0$, a finite subset $\mathcal{P} \subset K(A)$, a finite subset $\mathcal{H}_2 \subset A_{s.a.}$ and $\sigma > 0$ satisfying the following: Suppose that $L_1, L_2 : A \to M_k$ (for some integer $k \geq 1$) are two unital $\mathcal{G}$-$\delta$-multiplicative contractive completely positive linear maps such that

\[
[L_1]|_\mathcal{P} = [L_2]|_\mathcal{P},
\]

\[
\text{tr} \circ L_1(h) \geq \Delta(\hat{h}), \quad \text{tr} \circ L_2(h) \geq \Delta(\hat{h}) \quad \text{for all } h \in \mathcal{H}_1
\]

and

\[
|\text{tr} \circ L_1(h) - \text{tr} \circ L_2(h)| < \sigma \quad \text{for all } h \in \mathcal{H}_2,
\]

then there exists a unitary $u \in M_k$ such that

\[
\|\text{Ad} u \circ L_1(f) - L_2(f)\| < \epsilon \quad \text{for all } f \in \mathcal{F}. \quad (e 10.103)
\]
It follows from a combination of Lemma 2.14 and Lemma 2.6.